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SPECTRAL INVARIANT OF THE ZETA FUNCTION OF THE LAPLACIAN ON S^{4r-1}

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(*Dedicated to Professor T. S. Bhanu Murthy on his 60th Birthday*)

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The aim of this paper is to compute a spectral invariant of the zeta function $\zeta(\Delta, s)$ at $s = 0$ of the Laplace Beltrami operation Δ acting on forms of degree 2 on $(4r - 1)$ dimensional sphere S^{4r-1}.

1. INTRODUCTION

Let A be an elliptic pseudo-differential operator of positive order m on a compact n-dimensional manifold X. If A is self adjoint and positive, its eigen values are $\lambda > 0$. We define its Zeta function by

$$\zeta(A, s) = \text{Trace } A^{-s} = \sum_{\lambda>0} \lambda^{-s} \text{ (Atiyah } et\ al.^{1}\text{)}.$$

Here each eigen value is repeated as many times as its multiplicity. This series converges for $\text{Re}(s) > \frac{n}{m}$ (Atiyah *et al.*[2]) and hence gives a holomorphic function of the complex variable s in this half plane. Also $\zeta(A, s)$ can be analytically continued to the whole s-plane as a meromorphic function with simple poles. On the other hand, for any elliptic self adjoint pseudo differential operator B of positive order m on a compact manifold X, we define its eta function as

$$\eta(B, s) = \sum_{\lambda\neq 0} \text{Sign } \lambda \mid \lambda \mid^{-s}.$$

Here also λ runs over the eigen values of B and each eigen value repeats as many times as its multiplicity. But here as the operator B is not positive, its eigen values may be positive or negative and hence each eigen value is taken with its sign. When the operator B is also positive, we have

$$\zeta(B, s) = \eta(B, s).$$

The real valued invariants of the metric satisfying the condition that it is a continuous function of the metric can be obtained by evaluating ζ at some point s where it is known to be finite. For positive self adjoint elliptic operators, the zeta function defined above have finite values at $s = 0$ by the results in Seeley[5]. Atiyah *et al.*[2], gave two methods to prove the finiteness of η at $s = 0$ for the special operator B_v^{ev}. Using

Cobordism theory one can prove the finiteness of $\eta(0)$ and by using invariant theory, $\eta(0)$ can be computed explicitly for B_{ν}^{ev}. But in this paper we show a general method of analytic continuation to compute $\zeta(\Delta, s)$ at $s = 0$ for the Laplacian Δ and compute it for Δ acting on 2-forms on S^{4r-1}. The importance of this paper lies in the analytic continuation that we have used for the function of the form $\sum_{n=1}^{\infty} g(n)\{f(n)\}^{-s}$, where $f(n)$ and $g(n)$ are two polynomials of finite degrees. The method of analytic continuation that we have used in this paper can be used for all similar series.

2. Eigenvalues of Δ on 2-forms on S^{4r-1}

When G is a compact connected Lie group and K is a closed subgroup, we consider the quotient space $M = G/K$. The Laplace Beltrami operator or Laplacian is $\Delta = d\delta + \delta d$, where d is exterior differentiation and δ is operator adjoint to d. Δ is a self adjoint and elliptic differential operator on $C^{\infty}(\Lambda^{p}M)$ for each p. Here $C^{\infty}(\Lambda^{p}M)$ is the vector space of smooth sections of the vector bundle $\Lambda^{p}M$, the pth exterior power of the complexified cotangent bundle of M. The set of eigen values of Δ on $C^{\infty}(\Lambda^{p}M)$ is a discrete set of real numbers.

When G is a compact semisimple Lie group and B is the killing form of the Lie algebra $\mathscr{G}$ of G, the Casimir element is $C = \sum_{1 \leqslant i, j \leqslant N} c^{ij} X_i . X_j$ where $\{X_1, \ldots, X_N\}$ is a basis of $\mathscr{G}$ and $c^{ij} = (B(X_i, X_j))^{-1}$. When (G, K) is a compact symmetric pair with a compact connected semisimple Lie group G, we have the Cartan decomposition

$$\mathscr{G} = k \oplus m,$$

where k is the Lie algebra of K and m is the orthogonal complement to k in $\mathscr{G}$ with respect to the Killing form. Restricting the Killing form sign changed to m, we get a G-invariant Riemannian metric on $M = G/K$ and $\Delta = -C$ (Ikeda and Taniguchi[3]). So using Proposition 2.2 of Ikeda and Taniguchi[3], the eigenvalues of Δ can be written for $S^n = SO(n+1)/SO(n)$. We are interested in the Laplacian Δ acting on 2-forms on S^{4r-1}. Let τ be a Cartan subalgebra of $\mathscr{G}$ and $\lambda_1, \ldots, \lambda_{2r}$ be linear forms on τ. Any dominant integral form Λ in $G = SO(4r)$ with respect to τ is uniquely expressed as

$$\Lambda = k_1 \lambda_1 + \ldots + k_{2r} \lambda_{2r},$$

where $k_1, \ldots, k_{2r}$ are integers such that

$$k_1 \geqslant k_2 \geqslant \ldots \geqslant | k_{2r} | .$$

Then using the theorem 4.2 of Ikeda and Taniguchi[3] the following theorem can be easily stated.

Theorem 1—If $p = 2$ and Λ_{ρ} is the highest weight of the irreducible representation ρ intervening in $C^{\infty}(\Lambda^2 S^{4r-1})$, then

$$\Lambda_{\rho} = (k+1)\lambda_1 + \lambda_2 \text{ or } (k+1)\lambda_1 + \lambda_2 + \lambda_3 \text{ for } r \geqslant 2.$$

Moreover with respect to $\frac{-1}{2(4r-2)}$ times the Killing form, the eigen values of Δ are given by

$$(k+2)(k+4r-2) \text{ for } \Lambda_\rho = (k+1)\lambda_1 + \lambda_2$$

and

$$(k+3)(k+4r-3) \text{ for } \Lambda_\rho = (k+1)\lambda_1 + \lambda_2 + \lambda_3.$$

Here k runs over all non-negative integers. Moreover the multiplicity μ_ρ of the above in $C^\infty(\Lambda^2 S^{4r-1})$ is exactly one in the above two cases.

The multiplicity of an eigen value ω is the product of the multiplicity μ_ρ and the dimension of the representation ρ. The dimension of the representation ρ can be computed using Weyl's formula :

$$\frac{\pi_{\alpha>0} < \Lambda_\rho + \delta, \alpha >}{\pi_{\alpha>0} < \delta, \alpha >}.$$

Here $\alpha > 0$ are the positive roots in $SO(4r)$ and δ is half the sum of the positive roots in $SO(4r)$.

Hence we get the following table :

Λ_ρ	Eigen value
$(k+1)\lambda_1 + \lambda_2$	$(k+2)(k+4r-2)$
$(k+1)\lambda_1 + \lambda_2 + \lambda_3$	$(k+3)(k+4r-3)$

Multiplicity

$$\frac{2(k+1)(k+4r-1)(k+2r)}{(4r-3)(k+2)} \binom{k+4r-3}{4r-4}$$

$$\frac{(k+1)(k+2r)(k+4r-1)(4r-3)}{(k+3)(k+4r-3)} \binom{k+4r-2}{4r-3}$$

Here k runs over all non-negative integers.

3. Zeta Function of the Laplacian Δ for S^{4r-1}

As we consider the positive operator Δ acting on 2-forms on S^{4r-1}, we get $\eta(\Delta, s) = \zeta(\Delta, s)$.

$$\zeta(\Delta, s) = \sum_{n=0}^{\infty} \frac{2(n+1)(n+4r-1)(n+2r)}{(4r-3)(n+2)} \binom{n+4r-3}{4r-4} \times \{(n+2)(n+4r-2)\}^{-s}$$

(equation continued on p. 410)

$$+ \sum_{n=0}^{\infty} \frac{(n + 1)(n + 2r)(n + 4r - 1) 4r - 3)}{(n + 3)(n + 4r - 3)}$$

$$\times \binom{n + 4r - 2}{4r - 3} \left\{ (n + 3)(n + 4r - 3) \right\}^{-s}$$

we have to compute $\zeta(\Delta, s)$ at $s = 0$ by a method of analytic continuation. We denote this value by $\zeta(\Delta, 0)$.

We explain below the details of the analytic continuation which we use to compute the spectral invariant of the zeta function for S^{4r-1}.

Let

$$S = \sum_{n=1}^{\infty} g(n) \{f(n)\}^{-s}.$$

Here $g(n)$ and $f(n)$ are two primitive polynomials of finite degrees in n. Let us assume that the degree of $f(n)$ be k. Then this series S has analytic continuation as

$$\sum_{n=1}^{\infty} g(n) \{ (n^k + P(n))^{-s} - n^{-ks} \} + \sum_{n=1}^{\infty} g(n) n^{-ks}$$

in the entire complex plane.

Here $f(n) = n^k + P(n)$, $P(n)$ being a polynomial of degree $k-1$.

Let

$$(\mathrm{I}) = \sum_{n \leqslant C} g(n) \{ (n^k + P(n))^{-s} - n^{-ks} \}$$

$$(\mathrm{II}) = \sum_{n > C} g(n) n^{-ks} \left\{ \left(1 + \frac{P(n)}{n^k}\right)^{-s} - 1 \right\}$$

$$(\mathrm{III}) = \sum_{n=1}^{\infty} g(n) n^{-ks}.$$

Here the positive number C is chosen such that

$$\left| \frac{P(n)}{n^k} \right| < 1.$$

Then S has analytic continuation in the entire complex plane with

$$S = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).$$

(I) is an entire function whose value at $s = 0$ is 0. In (II), $\left(1 + \frac{P(n)}{n^k}\right)^{-s}$ can be ex-

panded using binomial theorem because $\left| \frac{P(n)}{n^k} \right| < 1$. Hence

$$(\text{II}) = \sum_{n>C} g(n)\, n^{-ks} \left\{ -s \frac{P(n)}{n^k} + \frac{s(s+1)}{2} \left(\frac{P(n)}{n^k} \right)^2 - \dots \text{ to } \infty \right\}.$$

As $s \to 0$ we have $s\,\zeta(s+1) \to 1$, ζ being the o rdinary Riemann zeta function[8]. So when $S \to 0$, (II) gives some constants due to first few terms and all the other terms in (II) will tend to zero. Morever the sum in (II) taken over any finite rectangle tends to zero as $s \to 0$. Let $g(n)$ be a polynomial of degree q such that

$$g(n) = \sum_{i=0}^{q} a_i n^i \text{ where } a_i \text{ (for } 0 \leqslant i \leqslant q)$$

are constants with $a_q = 1$.

Then

$$(\text{III}) = \sum_{i=0}^{q} a_i\, \zeta(ks - i).$$

So when $s \to 0$, (III) will contribute some constants to $\zeta(\Delta, 0)$ and finally we get $\zeta(\Delta, 0)$.

We now compute the value $\zeta(\Delta, 0)$ for S^{4r-1}.

Let

$$\frac{2(n+1)(n+4r-1)(n+2r)}{(4r-3)(n+2)} \binom{n+4r-3}{4r-4} = \frac{2}{(4r-3)!} g_1(n)$$

where $g_1(n) = \sum_{i=0}^{4r-2} a_i n^i$ with $a_{4r-2} = 1$. Similarly

we assume that

$$\frac{(4r-3)(n+1)(n+2r)(n+4r-1)}{(n+3)(n+4r-3)} \binom{n+4r-2}{4r-3}$$

$$= \frac{1}{(4r-4)!} g_2(n)$$

where $g_2(n) = \sum_{i=0}^{4r-2} b_i n^i$ with $b_{4r-2} = 1$.

We also assume that

$$f_1(n) = (n+2)(n+4r-2)$$

and

$$f_2(n) = (n + 2)(n + 4r - 3).$$

So

$$\zeta(\Delta, s) = 2r(4r - 1) + \frac{2r}{3}(4r - 1)(4r - 2)$$

(at $s = 0$)

$$+ \left\{\frac{2}{(4r - 3)!}\right\}\left\{\sum_{n=1}^{\infty} g_1(n)\,(f_1(n))^{-s}\right\}$$

$$+ \frac{1}{(4r - 4)!}\left\{\sum_{n=1}^{\infty} g_2(n)\,(f_2(n)^{-s})\right\} \text{ (at } s = 0).$$

We first find the contribution to $\zeta(\Delta, 0)$ from

$$S_1 = \sum_{n=1}^{\infty} g_1(n)\,(f_1(n))^{-s}.$$

This series has analytic continuation in the entire complex plane as

$S_1 = (\text{I}) + (\text{II}) + (\text{III})$ where

$$(\text{I}) = \sum_{n=1}^{C} g_1(n)\left\{(n^2 + 4rn + 8r - 4)^{-s} - n^{-2s}\right\}$$

$$(\text{II}) = \sum_{n=C+1}^{\infty} g_1(n) n^{-2s}\left\{(1 + \frac{4rn + 8r - 4)^{-s}}{n^2} - 1\right\} \text{ and}$$

$$(\text{III}) = \sum_{n=1}^{\infty} g_1(n)\, n^{-2s}.$$

Here C is chosen such that $\left|\frac{4rn + 8r - 4}{n^2}\right| < 1.$

Remarks : When $r = 2\ (S^7)$, C can be chosen to be greater or equal to 9. When $r = 3(S^{11})$, C can be chosen to be greater or equal to 13 and so on.

(I) is an entire function whose value at $s = 0$ is 0.

$$(\text{II}) = \sum_{n=C+1}^{\infty} g_1(n)\, n^{-2s}\left\{ -s.\ \frac{4rn + 8r - 4}{n^2}\right.$$

(equation continued on p. 413)

$$+ \frac{s(s+1)}{2} \left(\frac{4rn + 8r - 4}{n^2}\right)^2 - \dots \text{ to } \infty \Big\}.$$

Let $P_1(n) = 4rn + 8r - 4$ and

$$g_1(n)(P_1(n))^t = \sum_{i=0}^{4r+t-2} a_{(t,i)}\, n^i \text{ for } 1 \leqslant t \leqslant 4r - 1.$$

Denoting by $(\mathrm{II})_i$, the ith term of (II), we get

$$(\mathrm{II})_1 = -s \sum_{n=C+1}^{\infty} g_1(n)\, P_1(n)\, n^{-(2s+2)}$$

$$(\mathrm{II})_1 \text{ (at } s = 0) = -s \sum_{i=0}^{4r-1} a_{(1,i)}\, \zeta(2s + 2 - i) \,(at\ s = 0)$$

$$= -\tfrac{1}{2}\, a_{(1,1)}.$$

Similarly

$$(\mathrm{II})_2 \text{ (at } s = 0) = \tfrac{1}{4}\, a_{(2,3)},$$

$$(\mathrm{II})_3 \text{ (at } s = 0) = -\tfrac{1}{6}\, a_{(3,5)} \text{ and so on.}$$

Finally

$$(\mathrm{II})_{4r-1} \text{ (at } s = 0) = \frac{(-1)^{4r-1}}{2(4r-1)}\, a_{(4r-1,\, 8r-3)}$$

and all other terms in (II) will tend to 0. Now

$$(\mathrm{III}) = \sum_{n=1}^{\infty} g_1(n)\, n^{-2s}$$

$$= \sum_{i=0}^{4r-2} a_i\, \zeta(2s - i)$$

$$(\mathrm{III}) \text{ (at } s = 0) = -\frac{a_0}{2} - \frac{a_1 B_1}{2} + \frac{a_3 B_2}{4}$$

$$- \dots + \frac{(-1)^{2r-1} a_{4r-3} B_{2r-1}}{4r - 2}.$$

Here $B_1, B_2, \dots, B_{2r-1}$ are Bernoulli's numbers[8].

Similarly to find the contribution to $\zeta(\Delta, 0)$ from

$$S_2 = \sum_{n=1}^{\infty} g_2(n)\, (f_2(n))^{-s},$$

we assume that

$$P_2(n) = 4rn + 12r - 9$$

and

$$g_2(n)\,(P_2(n))^t = \sum_{i=0}^{4r+t-2} b_{(t,i)}\, n^i \text{ for } 1 \leqslant t \leqslant 4r - 1.$$

So using the notations that we have used, the following theorem is completely established.

Theorem 2—A spectral invariant of the zeta function $\zeta(\Delta, s)$ at $s = 0$ of the Laplace Beltrami operator Δ acting on 2-forms on S^{4r-1} $(r \geqslant 2)$ is

$$\frac{2}{(4r-3)!}\left\{\sum_{t=1}^{4r-1} \frac{(-1)^t a_{(t,2t-1)}}{2t} + \sum_{t=1}^{2r-1} \frac{(-1)^t a_{2t-1} B_t}{2t} - \frac{a_0}{2}\right\}$$

$$+ \frac{1}{(4r-4)!}\left\{\sum_{t=1}^{4r-1} \frac{(-1)^t b_{(t,2t-1)}}{2t} + \sum_{t=1}^{2r-1} \frac{(-1)^t b_{2t-1} B_t}{2t} - \frac{b_0}{2}\right\}$$

$$+ \frac{2r}{3}(16r^2 - 1).$$

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INCOMPLETE BLOCK DESIGNS OBTAINED BY THE GENERALISED ROW-JUXTAPOSITION AND GENERALISED COLUMN-CONCATENATION OF INCIDENCE MATRICES

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We define the concepts of 'generalized row-juxtaposition' and 'generalized column-concatenation' of two matrices. By taking these matrices as incidence matrices of block designs, several new constructions of binary and n-ary ($n>2$) incomplete block designs have been obtained.

1. INTRODUCTION

Taking the clue from Helleseth and Van Tilborg[3], we have defined, in the following, the concepts of 'generalized row-juxtaposition' and 'generalized column-concatenation' of matrices. A reference to what we have termed 'generalised column-concatenation' has been made in Helleseth and Van Tilborg[3].

The generalized juxtaposition of the rows and concatenation of the columns of (0, 1)-matrices provide some new constructions of 2-PBIBDS, 3-PBIBDS, partially balanced n-ary designs and balanced block designs. Eleven such constructions have been given in section 3 through various theorems and corollaries.

2. DEFINITIONS

Let A and B be arbitrary matrices of orders $m \times n$ and $p \times q$ respectively, over a field F.

Let

$$\mathbf{r}_i \ ; \ 1 \leqslant i \leqslant m$$

and

$$\mathbf{a}_j \ ; \ 1 \leqslant j \leqslant n$$

denote respectively the row and column vectors of A. Let

$$\mathbf{s}_i \ ; \ 1 \leqslant i \leqslant p$$

and

$$\mathbf{b}_j \; ; \; 1 \leqslant j \leqslant q$$

denote the corresponding vectors of B.

Given the ordered pair (A, B) of matrices, the generalised row-juxtaposition of the pair (A, B), denoted $G(A \mid B)$, is a function that juxtaposes all possible pairs of rows, one each from A and B taken in that order, giving rise to a new matrix $G(A \mid B)$ of order $mp \times (n + q)$:

$$G(A \mid B) = \left(\begin{array}{ccc|ccc|c|ccc} \mathbf{r}_1^T & \cdots & \mathbf{r}_1^T & \mathbf{r}_2^T & \cdots & \mathbf{r}_2^T & \cdots & \mathbf{r}_m^T & \cdots & \mathbf{r}_m^T \\ \mathbf{s}_1^T & \cdots & \mathbf{s}_p^T & \mathbf{s}_1^T & \cdots & \mathbf{s}_p^T & & \mathbf{s}_1^T & \cdots & \mathbf{s}_p^T \end{array} \right)^T.$$

Analogously, given the ordered pair (A, B), the generalized column-concatenation of (A, B), denoted $G(A/B)$ concatenates all possible pairs of columns, one each from A and B taken in that order, giving rise to a new matrix $G(A/B)$ of order $(m + p) \times nq$.

$$G(A/B) = \left(\begin{array}{ccc|ccc|c|ccc} \mathbf{a}_1 & \cdots & \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \cdots & \mathbf{a}_n \\ \mathbf{b}_1 & \cdots & \mathbf{b}_q & \mathbf{b}_1 & \cdots & \mathbf{b}_q & & \mathbf{b}_1 & \cdots & \mathbf{b}_q \end{array} \right).$$

The following standard abbreviations have been used:

SBIBD	:	symmetric balanced incomplete block design
n-PBIBD	:	n-associate class partially balanced incomplete block design
BBD	:	balanced block design
SRGDD	:	semi-regular group divisible design
S-PBnD	:	S-associate class partially balanced n-ary design
$C_p(A)$	:	Hasse-Minkowski invariant of A
$A(v, b, r, k, \lambda)$	:	BIBD with parameters v, b, r, k, λ having A as its incidence matrix.

3. Main Results

The following theorems can all be proved by the actual construction of the matrices and computation of the design parameters indicated. Most of the proofs are straight-forward, hence omitted.

Theorem 3.1—$G(A \mid B)$, where $A(v_1, b_1, r_1, k_1, \lambda_1)$ and $B(v_2, b_2, r_2, k_2, \lambda_2)$ are BIBDs, is a 3-PBIBD with rectangular association scheme, two unequal block sizes and the following parameters:

$$V = v_1 v_2, \; B = b_1 + b_2, \; R = r_1 + r_2;$$

$$K_1 = k_1 v_2,\ K_2 = k_2 v_1;$$

$$\Lambda_1 = r_1 + \lambda_{12},\ \Lambda_2 = r_2 + \lambda_1,\ \Lambda_3 = \lambda_1 + \lambda_2$$

$$n_1 = v_2 - 1,\ n_2 = v_1 - 1,\ n_3 = (v_1 - 1)(v_2 - 1)$$

there being b_1 blocks of size K_1 and b_2 blocks of size K_2.

Corollary 3.1.1—In the above construction, when $A = B$, $G(A \mid A)$ represents a 2-PBIBD with L_2-association scheme, with parameters

$$V = v^2,\ B = 2b,\ R = 2r,\ K = vk;$$

$$\Lambda_1 = r + \lambda,\ \Lambda_2 = 2\lambda;$$

$$n_1 = 2(v - 1),\ n_2 = (v - 1)^2.$$

Raghavarao[4] (p. 159) has given a construction for a 2-PBIBD with the above parameters, wherein he is required to use a BIBD and also an association scheme of Latin square type for constructing this 2-PBIBD. Our construction requires only a BIBD $A(v, b, r, k, \lambda)$ and nothing else.

Corollary 3.1.2—Each submatrix of $G(A \mid B)$ with constant column sum constitutes a 2-PBIBD of the singular group divisible type. The parameters of these designs are

$$(v_1 v_2,\ b_1, r_1,\ v_2 k_1\ ;\ r_1, \lambda_1\ ;\ v_2 - 1,\ (v_1 - 1) v_2)$$

and

$$(v_1 v_2,\ b_2,\ r_2,\ v_1 k_2\ ;\ r_2, \lambda_2\ ;\ v_1 - 1,\ (v_2 - 1) v_1)$$

respectively.

Since

$$G(A^T \mid B^T)^T = G(A/B),$$

we also have

Corollary 3.1.3—The transpose of $G(A/B)$ is a 3-PBIBD with rectangular association scheme and unequal block sizes, provided that $A(v_1, b_1, r_1, k_1, \lambda_1)$ and $B(v_2, b_2, r_2, k_2, \lambda_2)$ are linked block designs with respective block intersection numbers μ_1 and μ_2. Its parameters are

$$V = b_1 b_2,\ B = v_1 + v_2,\ R = k_1 + k_2\ ;\ K_1 = b_2 r_1,\ K_2 = b_1 r_2$$

$$\Lambda_1 = k_1 + \mu_2,\ \Lambda_2 = k_2 + \mu_1,\ \Lambda_3 = \mu_1 + \mu_2\ ;\ n_1 = b_2 - 1$$

$$n_2 = b_1 - 1,\ n_3 = (b_1 - 1)(b_2 - 1)\ .$$

Corollary 3.1.4—If $A = B$ in Corollary 3.1.3, then a 2-PBIBD with L_2-association scheme can always be constructed, whose parameters are $V = b^2$, $B = 2v$, $R = 2k$, $K = br$; $\Lambda_1 = k + \mu$, $\Lambda_2 = 2\mu$; $n_1 = 2(b - 1)$, $n_2 = (b - 1)^2$.

Theorem 3.2—If $A(v_1, b_1, r_1, k_1, \lambda_1)$ and $B(v_2, b_2, r_2, k_2, \lambda_2)$ are BIBDS, then $G(A/B)$ is a BBD with two different numbers of replicates as defined by Agrawal[1]. Its parameters are

$$V = v_1 + v_2,\ B = b_1 b_2;\ R_1 = b_2 r_1,\ R_2 = b_1 r_2$$

$$K = k_1 + k_2;\ \Lambda_1 = b_2 \lambda_1,\ \Lambda_2 = b_1 \lambda_2,\ \Lambda_{12} = r_1 r_2.$$

This result is corroborated by the structure of $G(A/B)$. $[G(A/B)]^T$, which is

$$\left[\begin{array}{c|c} b_2 \cdot AA^T & r_1 r_2 \cdot E_{v_1 v_2} \\ \hline r_1 r_2 \cdot E_{v_2 v_1} & b_1 \cdot BB^T \end{array}\right].$$

Corollary 3.2.1—When $A = B$ in the above theorem, the BBD becomes a SRGDD with parameters

$$V = 2v,\ B = b^2,\ R = rb,\ K = 2k$$

$$\Lambda_1 = b\lambda,\ \Lambda_2 = r^2;\ n_1 = v - 1,\ n_2 = v.$$

Corollary 3.2.2—The transpose of $G(A \mid B)$ is a BBD with two different numbers of replicates, provided that A and B are both linked block designs with respective block intersection numbers μ_1 and μ_2. Its parameters are

$$V = b_1 + b_2,\ B = v_1 v_2;\ R_1 = v_2 k_1;\ R_2 = v_1 k_2;\ K = r_1 + r_2$$

$$\Lambda_1 = v_2 \mu_1,\ \Lambda_2 = v_1 \mu_2,\ \Lambda_{12} = k_1 k_2.$$

This result is corroborated by the form of the matrix $[G(A \mid B)]^T$. $G(A \mid B)$, which is

$$\left[\begin{array}{c|c} v_2 \cdot A^T A & k_1 k_2 E_{b_1 b_2} \\ \hline k_1 k_2 \cdot E_{b_2 b_1} & v_1 \cdot B^T B \end{array}\right].$$

Corollary 3.2.3—If $A = B$ in Corollary 3.2.2, $[G(A \mid A)]^T$ represents a SRGDD with parameters

$$V = 2b,\ B = v^2,\ R = vk,\ K = 2r;\ \Lambda_1 = v\mu,\ \Lambda_2 = k^2$$

$$n_1 = v - 1,\ n_2 = v.$$

If $A_{m,n}$ and $B_{p,q}$ are any two matrices over F, then the product $G(A \mid B)$ $G(A/B)$ is defined if and only if $m + p = n + q$ and the product $G(A/B)\,G(A \mid B)$ is defined if and only if

$$mp = nq.$$

If A and B are the incidence matrices of BIBDs, then a necessary and sufficient condition for the existence of any or both of the above product—matrices is that A and B are SBIBDSs.

In what follows, unless stated otherwise, $A(v_1, k_1, \lambda_1)$ and $B(v_2, k_2, \lambda_2)$ will be assumed to be SBIBDs.

It has been shown[5] that the Kronecker sum of incomplete block designs gives ternary designs. We obtain below, an n-ary design as the Kronecker sum of the squares of two SBIBDs.

Theorem 3.3—The product

$$G(A \mid B)\, G(A/B)$$

is the Kronecker sum of A^2 and B^2 and is the incidence matrix of a 3-PBnD with rectangular association scheme, where $n = k_1 + k_2 + 1$. The parameters of the design are

$$V = v_1 v_2 = B, \quad R = v_1 k_2^2 + v_2 k_1^2 = K$$

$$\Lambda_1 = v_2 k_1 ((k_1 - 1)\lambda_1 + k_1) + v_1 \lambda_2 (k_2^2 + k_2 - \lambda_2) + 2k_1^2 k_2^2$$

$$\Lambda_2 = v_2 \lambda_1 (k_1^2 + k_1 - \lambda_1) + v_1 k_2 ((k_2 - 1)\lambda_2 + k_2) + 2k_1^2 k_2^2$$

$$\Lambda_3 = v_2 \lambda_1 (k_1^2 + k_1 - \lambda_1) + v_1 \lambda_2 (k_2^2 + k_2 - \lambda_2) + 2k_1^2 k_2^2$$

$$n_1 = v_2 - 1, \; n_2 = v_1 - 1, \; n_3 = (v_1 - 1)(v_2 - 1).$$

In the above theorem, as well as in the ones that follow, we have extended the definition of n-ary designs given by Das and Rao[3] to include those designs for which incidence matrices have any of n distinct non-negative integers as its elements. The entries in the matrix $G(A \mid B)\, G(A/B)$ above belong to the set $\{0, 1, \ldots, k_1 + k_2\}$.

Corollary 3.3.1—In the above theorem, if $A = B$, then a 2-PBnD can always be constructed, where $n = 2k + 1$. Its parameters are

$$V = v^2 = B, \; R = 2vk^2 = K$$

$$\Lambda_1 = v\lambda (k^2 + k - \lambda) + vk^3 + 2k^4$$

$$\Lambda_2 = 2v\lambda (k^2 + k - \lambda) + 2k^4$$

$$n_1 = 2(v - 1), \; n_2 = (v - 1)^2.$$

The above design has L_2-association scheme.

Agrawal[1] has defined binary BBDs with constant block sizes. In the following theorem, we obtain an n-ary BBD with unequal block sizes.

Theorem 3.4—The matrix $G(A/B)\, G(A \mid B)$

$$= \left[\begin{array}{c|c} \dfrac{v_2 A^2}{k_1 k_2 E_{v_2 v_1}} & \dfrac{k_1 k_2 E_{v_1 v_2}}{v_1 B^2} \end{array} \right]$$

is the incidence matrix of an n-ary BBD with two different numbers of replications and two different block sizes. The parameters of this design are

$$V = v_1 + v_2 = B$$

$$R_1 = v_2 k_1 (k_1 + k_2) = K_1$$

$$R_2 = v_1 k_2 (k_1 + k_2) = K_2$$

$$\Lambda_1 = \lambda_1 v_2^2 (k_1^2 + k_1 - \lambda_1) + k_1^2 k_2^2 v_2$$

$$\Lambda_2 = \lambda_2 v_1^2 (k_2^2 + k_2 - \lambda_2) + k_1^2 k_2^2 v_1$$

$$\Lambda_{12} = k_1 k_2 (v_2 k_1^2 + v_1 k_2^2).$$

Here

$$n = k_1 + k_2 + 2.$$

Corollary 3.4.1—In the above theorem, replacing B by A we get a group divisible 2-PBnD, where $n = k + 2$, with the following parameters :

$$V = 2v = B, \; R = 2vk^2 = K$$

$$\Lambda_1 = \lambda v^2 (k^2 + k - \lambda) + k^4 v, \; \Lambda_2 = 2vk^4$$

$$n_1 = v - 1, \; n_2 = v.$$

Theorem 3.5—If A, B, C and D are four linked BIBDs with parameters respectively

$$(v_i, b_i, r_i, k_i, \lambda_i, \mu_i), \; i = 1, 2, 3, 4$$

such that

$$v_1 = b_3$$

and

$$v_2 = b_4$$

then

$$E = [G (A \mid B) \;\; | \;\; [G (C/D)]^T]$$

is a rectangular 3-PBIBD with four unequal block sizes. Its parameters are

$$V = v_1 v_2, \; B = 2 (v_1 + v_2), \; R = r_1 + r_2 + k_3 + k_4$$

$$K_1 = v_2 k_1, \; K_2 = v_1 k_2, \; K_3 = v_2 r_3, \; K_4 = v_1 r_4$$

$$\Lambda_1 = r_1 + \lambda_2 + k_3 + \mu_4, \; \Lambda_2 = \lambda_1 + r_2 + \mu_3 + k_4,$$

$$\Lambda_3 = \lambda_1 + \lambda_2 + \mu_3 + \mu_4$$

$$n_1 = v_2 - 1, \; n_2 = v_1 - 1, \; n_3 = (v_1 - 1)(v_2 - 1).$$

Corollary 3.5.1—Under the condition that $A = B = C = D$, the matrix $F = E^T$ (where E is as given in Theorem 3.5) is a 3-PBIBD with Extended Group divisible association scheme and parameters given by

$$V = 4v, \; B = v^2, \; R = kv, \; K = 4k$$

$$\Lambda_1 = v\lambda, \; \Lambda_2 = k^2, \; \Lambda_3 = vk$$

$$n_1 = 2(v - 1), \; n_2 = 2v, \; n_3 = 1.$$

Lemma 3.1—Suppose that A, B, C and D are BIBDs with respective sets of parameters

$$(v_i, b_i, r_i, k_i, \lambda_i), \; i = 1, 2, 3, 4.$$

(i) If $N = G(A \mid B)$, then $NN^T = AA^T \circledS BB^T$

(ii) If $N = [G(A/B)]^T$, then $NN^T = A^TA \circledS B^TB$

(iii) If $N = [G(A \mid B) \mid [G(A/B)]^T]$

then

$$NN^T = (AA^T + C^TC) \circledS (BB^T + D^TD)$$

when

$$v_1 = b_3 \text{ and } v_2 = b_4.$$

where $\circledS$ denote the Kronecker sum of matrices.

The above lemma may be useful in determining $C_p(NN^T)$ and hence finding necessary conditions for the existence of the designs denoted by N.

4. Summary

The following table summarizes the results obtained in the last section.

Sr. No.	Source	Operation	Result
1.	Thm. 3.1	$G(A \mid B)$	3-PBIBD with rect. asso. Sch. and 2 unequal block sizes.
2.	Cor. 3.1.1	$G(A \mid A)$	2-PBIBD with L_2 asso. sch.

(*Table continued on p.* 422)

Sr. No.	Source	Operation	Result
3.	Cor. 3.1.2	Submatrix of $G\,(A \mid B)$	Singular GDD
4.	Thm. 3.2	$G\,(A/B)$	BBD with two replicates.
5.	Cor. 3.2.1	$G\,(A/A)$	SRGDD
6.	Thm. 3.3	$G\,(A \mid B)\,G\,(A/B) = A^2 \circledast B^2$	3-PBnD with rect. asso. sch.
7.	Cor. 3.3.1	$G\,(A \mid A)\,G\,(A/A)$	2-PBnD with L_2-asso. sch.
8.	Thm. 3.4	$G\,(A/B)\,G\,(A \mid B)$	n-ary BBD with two replicates
9.	Cor. 3.4.1	$G\,(A/A)\,G\,(A \mid A)$	GD-PBnD.
10.	Thm. 3.5	$\{G\,(A \mid B) \mid [G\,(C/D)]^T\}$	3-PBIBD with rect. asso. sch. and 4 block sizes.
11.	Cor. 3.5.1	$\{G\,(A \mid A) \mid [G\,(A/A)]^T\}$	3-PBIBD with extended GD asso. Sch.

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ON C 3-LIKE FINSLER SPACES

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In this paper we have studied the properties of Finsler space F^n of dimension $n \geqslant 4$ in which C_{ijk} is of the form $C_{ijk} = \mathscr{C}_{(ijk)} \{h_{ij}\, a_k + C_i\, C_j\, b_k\}$. Since C_{ijk} of any three dimensional Finsler space is of the above form, any n-dimensional Finsler space F^n ($n \geqslant 4$) will be called a $C3$-like Finsler space if its C_{ijk} is of this from.

1. Introduction

There are three kinds of torsion tensor in Cartan's theory of Finsler space F^n. Two of them are (h) hv-torsion tensor C_{ijk} and (v) hv-torsion tensor P_{ijk} which are symmetric in all its indices. It is obvious that F^n is Riemannian if the tensor C_{ijk} vanishes and Deicke's theorem shows that F^n is also Riemannian even if the tensor $C_i \;(=g^{jk}\, C_{ijk})$, contracted by the reciprocal tensor g^{jk} of the fundamental tensor g_{jk}, vanishes[3,5,14,17]. As is well known, in the 2-dimensional Finsler space F^2, C_{ijk} is always written in the form

$$LC_{ijk} = I m_i m_j m_k \qquad \ldots(1.1)$$

where $L(x,y)$ is the fundamental function, m_i is unit vector orthogonal to the element of support y^i and I is called the principal scalar by Berwald[1,2,17]. A Moor studied the property of n-dimensional Finsler space F^n ($n \geqslant 3$) in which C_{ijk} is of the form (1.1)[12,13]. In this case, however the third curvature tensor S_{ijkl} vanishes, so that F^n is reduced to a Riemannian space in virture of Brickell's theorem provided that the metric is symmetric[4,18].

In a 3-dimensional Finsler space F^3, C_{ijk} is always written in the form

$$L\, C_{ijk} = H m_i m_j m_k - J\, \mathscr{C}_{(ijk)} \{m_i m_j n_k\} + I\, \mathscr{C}_{(ijk)} \{m_i n_j n_k\} + J\, n_i n_j n_k \qquad \ldots(1.2)$$

where $\mathscr{C}_{(ijk)}\{\ \}$ denote the cyclic permutation of indices i, j, k and addition. H, I and J are main scalars and (l_i, m_i, n_i) is Moor's frame[7,9,15]. Here $l_i = \dot{\partial}_i L$ is the unit vector along the element of support, m_i is the unit vector along C_i i.e. $m_i = C_i/C$ where $C^2 = g^{ij} C_i C_j$ and n_i is a unit vector orthogonal to the vectors l_i and m_i. Since the angular metric tensor h_{ij} in F^3 can be written as

$$h_{ij} = m_i m_j + n_i n_j, \quad \text{...(1.3)}$$

we may write (1.2) as

$$C_{ijk} = \mathscr{C}_{(ijk)}\{h_{ij}\, a_k + C_i C_j b_k\} \quad \text{...(1.4)}$$

where

$$a_k = \frac{1}{L}\left(I\, m_k + \frac{J}{3} n_k \right)$$

and

$$b_k = \frac{1}{LC^2}\left\{\left(\frac{H}{3} - I\right) m_k + \frac{4J}{3} n_k \right\}.$$

Many workers[6,10,11,16] have obtained various interesting special forms of C_{ijk}. They are

Generalized C-reducible :

$$C_{ijk} = \mathscr{C}_{(ijk)}\{P_{ij}\, Q_k\}. \quad \text{...(1.5)}$$

Quasi C-reducible :

$$C_{ijk} = \mathscr{C}_{(ijk)}\{P_{ij}\, C_k\} \quad \text{...(1.6)}$$

Semi C-reducible :

$$C_{ijk} = \frac{P}{n+1}\, \mathscr{C}_{(ijk)}\left\{h_{ij}\, C_k\right\} + \frac{q}{C^2}\, C_i C_j C_k \quad \text{...(1.7)}$$

C-reducible :

$$C_{ijk} = \frac{1}{n+1}\, \mathscr{C}_{(ijk)}\{h_{ij}\, C_k\} \quad \text{...(1.8)}$$

$C2$-like :

$$C_{ijk} = \frac{1}{C^2}\, C_i C_j C_k \quad \text{...(1.9)}$$

where P_{ij} in (1.5) and (1.6) is a symmetric tensor, Q_i in (1.5) is a covariant vector such that $P_{i0} = Q_0 = 0$ and p, q in (1.7) are scalars satisfying $p + q = 1$.

The purpose of this paper is to study the properties of F^n of dimension $n \geqslant 4$ in which C_{ijk} is of the form (1.4). Since C_{ijk} of any three dimensional Finsler space is

of the form (1.4), an n-dimensional Finsler space F^n ($n \geqslant 4$) will be called a $C3$-like Finsler space if its C_{ijk} is of the form (1.4).

2. The Properties of $C3$-like Finsler Spaces

Definition 2.1— A Finsler space F^n of dimension $n \geqslant 4$ is called a $C3$-like Finsler space if there exist covariant vector fields a_k and b_k in F^n such that its (h) hv-torsion tensor C_{ijk} can be written as

$$C_{ijk} = \mathscr{C}_{(ijk)} \{h_{ij}\, a_k + C_i C_j b_k\}. \qquad \ldots(2.1)$$

Since C_{ijk} is an indicatory tensor it follows that a_k and b_k are indicatory tensor i.e. $a_0 = b_0 = 0$. If b_k is a null vector then contracting (2.1) with g^{jk} and putting $b_k = 0$ we get $a_k = \frac{1}{n+1} C_k$. Hence it reduces to a C-reducible Finsler space. Thus a $C3$-like Finsler space is a generalization of a C-reducible Finsler space. Again if a_k is a null vector then contraction of (2.1) with g^{jk} gives

$$C_i = C^2\, b_i + 2b_j\, C^j\, C_i \qquad \ldots(2.2)$$

which after contraction with C^i gives $b_j\, C^j = \frac{1}{3}$. Thus (2.2) gives $b_i = \frac{C_i}{3C^2}$.

Hence for $a_k = 0$, the form (2.1) of C_{ijk} reduce to

$$C_{ijk} = \frac{1}{C^2}\, C_i C_j C_k.$$

Therefore the $C3$-like Finsler space is also a generalization of $C2$-like Finsler space. Furthermore if we put $a_k = PC_k/n+1$, and $b_k = qC_k/3C^2$ then the form (2.1) reduces to the form (1.7) i.e. F^n becomes a semi C-reducible Finsler space. Thus a $C3$-like Finsler space is also a generalization of semi-C-reducible Finsler space. Contracting (2.1) with g^{jk} we get

$$KC_i = (n + 1)\, a_i + C^2\, b_i \qquad \ldots(2.3)$$

where $K = 1 - b_j c^j$. If $b_j c^j = 1$, then $K = 0$ and hence from (2.3) we get $a_i = -\frac{C^2}{n+1}\, b_i$.

Substituting this value of a_i in (2.1) we get

$$C_{ijk} = \mathscr{C}_{(ijk)} \{P_{ij}\, b_k\} \qquad \ldots(2.4)$$

where

$$P_{ij} = C_i C_j - \frac{C^2}{n+1}\, h_{ij}.$$

The form (2.4) is the form (1.5). A Finsler space whose C_{ijk} is of the form (1.5) is called generalized C-reducible Finsler space[16]. The examples of non-Riemannian generalized C-reducible Finsler space is given by Okada and Numata[16]. If

$$\alpha = \{a_{ij}\,(x)\, y^i y^j\}^{1/2}$$

and

$$\beta = \{b_{ij}(x)\, y^i y^j\}^{1/2}$$

be two independent functions then an n ($\geqslant 2$) dimensional Finsler space F^n with a metric $L = L(\alpha, \beta)$ is generalized C-reducile.

Hence we have the following :

Theorem 2.1—Every $C3$-like Finsler space is a generalized C-reducible Finsler space provided $b_i C^i = 1$.

If $b_i C^i \neq 1$ then from (2.1) and (2.3) we get

$$C_{ijk} = \mathscr{C}_{(ijk)}\{Q_{ij}\, a_k\} + d_i d_j d_k \qquad \text{...(2.5)}$$

where

$$Q_{ij} = h_{ij} + \tfrac{1}{2}\left(\frac{n+1}{K}\right)^2 (a_i b_j + a_j b_i) + \frac{2(n+1)C^2}{K^2} b_i b_j \qquad \text{...(2.6)}$$

and

$$d_i = \left(\frac{3\,C^4}{K^2}\right)^{1/3} b_i.$$

Now we shall give below an example of a Finsler space whose (h) hv-torsion tensor C_{ijk} is of the form (2.5). Let F^n be a C-reducible Finsler space with metric function $L(x, y)$. If F^{*n} be another Finsler space whose metric function $L^*(x, y)$ is given by

$$L^{*3}(x, y) = L^3(x, y) + (b_i y^i)^3$$

where $b_i(x)$ is one form in F^n. The (h) hv-torsion tensor $\overset{*}{C}_{ijk}$ of F^{*n} and those of F^n are related by

$$C^*_{ijk} = P\, C_{ijk} + \frac{p\, q^2}{2L}(h_{ij} m_k + h_{ik} m_j + h_{jk} m_i) - \frac{2p^3 - 1}{L} p\, m_i m_j m_k \qquad \text{...(2.7)}$$

where

$$m_i = q\, l_i - p\, b_i,\ q = \frac{L}{L^*},\ q = \frac{\beta}{L^*},\ \beta = b_i y^i.$$

Thus putting

$$C_{ijk} = \frac{1}{n+1}(h_{ij} C_k + h_{ik} C_j + h_{jk} C_i) \text{ in (2.7)}$$

we get

$$C^*_{ijk} = \mathcal{C}_{(ijk)}\{Q_{ij}\, a_k\} + d_i d_j d_k$$

where

$$Q_{ij} = p\, h_{ij},\ a_i = \frac{C_i}{n+1} + \frac{q^2}{2L} m_i \text{ and } d_i = -\left(\frac{2p^3-1}{L}\, p\right)^{\frac{1}{3}} m_i.$$

A Randers space is a Finsler space whose metric funciion $L(x, y)$ is of the form

$$L(x, y) = (a_{ij}(x)\, y^i y^j)^{1/2} + f_i(x)\, y^i$$

where $a_{ij}(x)$ is a Riemannian metric tensor and $f_i(x)$ is one form.

A Kropina space is a Finsler space in which

$$L(x, y) = \frac{(a_{ij}(x)\, y^i y^j)^{1/2}}{f_i(x)\, y^i}.$$

It is well known[8,19] that the Randers space and the Kropina space are C-reducible Finsler spaces. Thus if the metric function $L^*(x, y)$ of F^{*n} is given by

$$L^{*3}(x, y) = \{(a_{ij}(x)\, y^i y^j)^{1/2} + f_i(x)\, y^i\}^3 + (b_i\, y^i)^3$$

or

$$L^{*3}(x, y) = \frac{(a_{ij}(x)\, y^i\, y^j)^{3/2}}{(f_i(x)\, y^i)^3} + (b_i\, y^i)^3$$

then the (h) hv-torsion tensor of F^{*n} is of the form (2.5).

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SOME SERIES SOLUTIONS OF THE DUFFING EQUATION

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Some approximate solutions of the Duffing equation is considered. The approximate solutions are exact for a particular initial value.

1. INTRODUCTION

Many physical phenomena are modelled by ordinary nonlinear differential equations. In order to analyse the behaviour of the physical systems, we need the solutions of the differential equations which is extremely difficult in the nonlinear case. However there are methods to obtain approximate solutions of nonlinear equations. Peters[1] and Usher[3] have obtained some approximate solutions to the equation

$$\ddot{x}+\omega^2 x= -\alpha x^2 \qquad \ldots(1)$$

subject to the initial condition

$$x\,(o) = A,\ \dot{x}\,(o) = 0 \qquad \ldots(2)$$

But their approximate solution is not equal to

$$x\,(t) = -\omega^2/\alpha \text{ when } A = -\omega^2/\alpha.$$

Such a difficulty has been eliminated by Shidfar and Sadeghi[2] by using a different method.

In this paper we shall obtain some approximate solutions of the Duffing equation

$$\ddot{x} + \omega^2 x = \beta^2 x^3 \qquad \ldots(3)$$

with initial conditions $x\,(0) = A$ and $\dot{x}\,(0) = 0$

by suitably adopting the method discussed in Shidfer and Sadeghi.[2]

In pursuit of this we put

$$x = u - \omega/\sqrt{3}\,\beta. \qquad \ldots(4)$$

Inserting (4) into (2) and (3) we obtain

$$\ddot{u} + \sqrt{3}\,\omega\beta\, u^2 - \beta^2 u^3 = 2\,\omega^3/3\sqrt{3}\,\beta \qquad ...(5)$$

$$u(o) = A + \frac{\omega}{\sqrt{3}\,\beta}\,,\ \dot{u}(o) = O \qquad ...(6)$$

Next we shall discuss the series solution of equation (5).

2. Series Solution

To solve equation (5) let us investigate the solution in the form

$$u(t) = c_0 + c_1 \sin \omega t + c_2 \sin^2 \omega t + \ldots \qquad ...(7)$$

where the coefficients c_0, c_1, c_2 ... are constants and to be determined.

Substituting (7) into $u(o) = A + \omega/\sqrt{3}\beta$ we get

$$c_0 = A + \omega/\sqrt{3}\beta. \qquad ...(8)$$

Using (7) into LHS of (5) and equating the sum of the coefficients of each $\sin^n \omega t$, $n = 0, 1, 2, \ldots$ to zero, we have

$$2\,\omega^2 c_2 + \sqrt{3}\,\omega\beta\; c_0^2 - \beta^2 c_0^3 = 2\,\omega^3/3\sqrt{3}\beta,\ n = 0$$

$$c_{n+2} = \frac{n^2}{(n+1)(n+2)}\, c_n - \frac{\sqrt{3}\beta}{\omega(n+1)(n+2)}\, b_n$$

$$+ \frac{\beta^2}{\omega^2(n+1)(n+2)}(b_0 c_n + \ldots + b_n c_0)\ n \geqslant 1 \qquad ...(9)$$

where $b_n = c_0 c_n + \ldots + c_n c_0\ b_0 = c_0^2$.

From $\dot{u}(o) = 0$ and (7) we can immediately concluded that $c_1 = 0$.

The second equality of (9) and $c_1 = 0$ yields $c_3 = 0$, $c_5 = 0$ ... successively. For the coefficients with even subscript we obtain

$$c_0 = A + \omega/\sqrt{3}\beta$$

$$c_2 = \frac{A\,\beta^2}{2\,\omega^2}(A^2 - \omega^2/\beta^2) \qquad ...(10)$$

$$c_4 = \frac{A\,\beta^4}{8\,\omega^4}(A^2 - \omega^2/\beta^2)(A^2 + \omega^2/\beta^2), \text{ etc.}$$

This gives the solution

$$x(t) = A + \frac{A\,\beta^2}{2\,\omega^2}(A^2 - \omega^2/\beta^2)\sin^2 \omega t + \ldots \qquad ...(11)$$

for equation (3) subject to the initial condition (2).

Remark 1 : As we see the coefficients c_2, c_4, vanish for $A = \pm\, \omega/\beta$ and it can be readily shown that the other coefficients also vanish for this value of A. Therefore $x(t) = \pm\, \omega/\beta$ for $A = \pm\, \omega/\beta$. Hence the solution is exact for this particular initial conditions.

3. Convergence of the Solution

It is interesting to note that the series solution (7) convergent for all t. Infact, in the case $c_0 > 0$, $c_2 > 0$ and $\beta = -\gamma$, we obtain from (9) $c_n > 0$ for $n = 4, 6, \ldots$. By the second equality of (9) we can immediately conclude that

$$\sum_{n=2}^{\infty} (n+1)(n+2)\, c_{n+2} = \sum_{n=2}^{\infty} n^2 c_n + \frac{\sqrt{3}\gamma}{\omega}\Big(\sum_{n=2}^{\infty} c_n\Big)^2 + \frac{\gamma^2}{\omega^2}\Big(\sum_{n=0}^{\infty} c_n\Big)^3 - \frac{\sqrt{3}\gamma}{\omega}\, c_0^2 - \frac{\gamma^2}{\omega^2}\, c_0^3 \qquad \ldots(12)$$

or

$$\sum_{n=0}^{\infty} n c_n + \frac{\sqrt{3}\beta}{\omega}\Big(\sum_{n=0}^{\infty} c_n\Big)^2 + \frac{\gamma^2}{\omega^2}\Big(\sum_{n=0}^{\infty} c_n\Big)^3 = \frac{\sqrt{3}\gamma}{\omega}\, c_0^2 + \frac{\gamma^2}{\omega^2}\, c_0^3 - 2\, c_2 \qquad \ldots(13)$$

Therefore, in this case, the series of coefficients of series solution (9) converges. Now if we put

$$c_0^1 = c_0, \quad c_1^1 = 0, \quad c_2^1 = c_2,$$

$$c_{n+2}^1 = \frac{n^2}{(n+1)(n+2)}\, c_n^1 + \frac{\sqrt{3}\,\beta\, b_n^1}{\omega\,(n+1)(n+2)} + \frac{\beta^2}{\omega^2 (n+1)(n+2)}\,(b_0^1\, c_n^1 + \ldots + b_n^1\, c_0^1) \qquad \ldots(14)$$

where

$$b_n^1 = c_0^1\, c_n^1 + \ldots + c_n^1\, c_0^1, \; n > 1.$$

then the series $\sum_{n=0}^{\infty} c_n^1$ converges. Since $|c_n| \leqslant c_n^1$ it follows that the series solution (7) is absolutely convergent for all t and consequently the series solution (7) converges for all t.

4. Power Series Solution

Finally we seek a power series solution in the form of power series say

$$u(t) = \sum_{n=0}^{\infty} a_n t^n \qquad \text{...(15)}$$

for equation (5) subject to the initial condition (6).

Inserting (15) into (5) and using the initial conditions (6) we can conclude

$$a_1 = 0, a_3 = 0, a_5 = 0 \, . \, . \qquad \text{...(16)}$$

and

$$a_0 = A + \omega/\sqrt{3}\,\beta$$

$$2\,a_2 + \sqrt{3}\,\omega\beta\, a_0^2 - \beta^2 a_0^3 = 2\,\omega^3/3\sqrt{3}\,\beta.$$

$$a_{n+2} = \frac{-\sqrt{3}\,\beta\omega}{(n+1)(n+2)}\, d_n + \frac{\beta^2}{(n+1)(n+2)}\,(d_0 a_n + d_2\, a_{n-2} + \, . \, .$$

$$+ d_n a_0)\; n = 2\,k,\, k \geqslant 1 \qquad \text{...(17)}$$

where

$$d_n = a_0 a_n + a_2\, a_{n-2} + \, . \, . + a_n\, a_o.$$

The coefficients (17) are vanished for $A = \pm\, \omega/\beta$ and the corresponding solution of eqn. (3) becomes

$$x(t) = \pm\, \omega/\beta.$$

Remark 2 : The solutions (7) and (15) are identical. This can be easily verified by substituting the Maclaurent series of $\sin \omega t$ into (11) and writing it as a power series.

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GENERALIZED ORTHOGONALITY RELATION FOR THE FLEXURE OF SECTORIAL PLATES

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In this paper using some refined plate theories a generalized orthogonality relation for the flexure of a sectorial plate is derived.

INTRODUCTION

Eigenfunctions of the biharmonic equation and orthogonality or biorthogonality relations for satisfying the boundary conditions have been extensively used in the two-dimensional elasticity theory and in the flexure of thin elastic plates. Smith[1] has presented an orthogonality relation for the biharmonic eigenfunctions associated with a rectangular strip whose long edges are stress free. Greenberg[2] has brought out some orthogonality relations satisfied by the biharmonic eigenfunctions associated with a rectangle whose long edges are stress free and a sector plate whose radial edges are stress free. Prokopov[3] has shown that the orthogonality relations obtained by Greenberg exist even ifthe long edges of the rectangle are built-in. A very general method of developing a biorthogonality relation for obtaining the solutions for a semi-infinite strip whose long edges are stress free is provided by Johnson and Little[4]. Little and Childs[5], Klemn and Little[6] and Klemn and Fernandes[7] have successfully used the method of biorthogonality relations to provide solutions of various boundary value problems for different geometries. And newer orthogonality relations for the boundary value problems associated with a rectangle or a cylinder or a curved beam have been discovered by Nuller[8], Kostarev and Prokopov[9], Fama[10] and a few others.

A study to develop eigenfunction solutions has not been made for the plate theory which takes into account shear deformation of the plate. Further, the available literature does not show the existence of generalized orthogonality relations satisfied by these functions. These relations are indeed very elegant and can be directly used to reduce boundary value problems to a system of linear algebraic equations. Accordingly the theme of eigenvalues, eigenfunctions and eigenfunction expansions in relation to the refined theory of plates has been explored in considerable detail in the doctoral thesis of Rao[11]. In what follows, a derivation of the generalized orthogonality relation for the flexure of a flat curved plate taking shear deformation into account is presented.

Basic Results

The basic governing differential equation in polar coordinates for the flexure of plates in the refined theories of Reissner[12], Hencky[13] and Mindlin[14] are

$$\Delta\Delta w = \frac{q}{D} - \frac{h^2}{6\eta^2}\left(\frac{1-n}{1-\mu}\right)\frac{1}{D}\Delta\Delta q \qquad \text{...(1a)}$$

$$\Delta\psi = \frac{12\eta^2}{h^2}\psi \qquad \text{...(1b)}$$

where

$$\epsilon_r = -\frac{\partial w}{\partial r} + \frac{h^2}{6\eta^2 D(1-\mu)} Q_r \qquad \text{...(2a)}$$

$$\epsilon_\theta = -\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{h^2}{6\eta^2 D(1-\mu)} Q_\theta \qquad \text{...(2b)}$$

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial Q^2}\right]\right.$$

$$+ \frac{h^3}{6\eta^2}\frac{\partial Q_r}{\partial r} - \frac{h^2}{6m\eta^2}\frac{q\mu}{1-\mu} \qquad \text{...(2c)}$$

$$M_\theta = -D\left[\mu\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right]$$

$$+ \frac{h^2}{6\eta^2}\left(\frac{Q_r}{r} + \frac{1}{r}\frac{\partial \theta_\theta}{\partial \theta}\right) - \frac{h^2}{6m\eta^2}\frac{qu}{1-\mu} \qquad \text{...(2d)}$$

$$M_{r\theta} = D(1-\mu)\left[\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right]$$

$$- \frac{h^2}{12\eta^2}\left[\frac{1}{r}\frac{\partial Q_r}{\partial \theta} - \frac{Q_\theta}{r} + \frac{\partial Q_\theta}{\partial r}\right] \qquad \text{...(2e)}$$

$$Q_r = -D\frac{\partial}{\partial r}(\Delta w) + \frac{1}{r}\frac{\partial \psi}{\partial \theta} - \frac{h^2}{6\eta^2}\frac{1-n}{1-\mu}\frac{1}{D}\frac{\partial q}{\partial r} \qquad \text{...(2f)}$$

$$Q_\theta = -D\frac{1}{r}\frac{\partial}{\partial \theta}(\Delta w) - \frac{\partial \psi}{\partial r} - \frac{h^2}{6\eta^2}\frac{1-n}{1-\mu}\frac{1}{D}\frac{\partial q}{\partial \theta}. \qquad \text{...(2g)}$$

Here η^2 is the shear rigidity factor introduced by Mindlin. For Reissner's theory we set $m = 2$, $n = \mu/2$ and $\eta^2 = 5/6$. For Hencky's theory we set $m = 1$, $n = 0$ and $\eta^2 = 1$. For Mindlin's theory we set $m = 1$, $n = 0$ and $\eta^2 = 5/6$.

Formulation of the Problem

We now assume that the curved plate occupying the region $a \leqslant r \leqslant b$, $0 \leqslant \theta \leqslant \delta$ is free from the load $q(r, \theta)$ and accordingly set $q = 0$ in the equations (1) and (2). Defining a new function

$$p = \Delta w \qquad \text{...(3)}$$

the plate equation (1a) may be written as

$$\Delta p = 0 \qquad \ldots(4)$$

(1b), (3) and (4) provide three partial differential equations of the second order. Alternatively we want to formulate three partial differential equations involving either the variables w, M_θ and ϵ_r or the variables Q_θ, ϵ_θ and $M_{r\theta}$. To this end we connect p and ψ with the above mentioned variables. It is shown in Appendix (A.1 – A.2) that

$$p = -\frac{M_\theta}{D} - (1-\mu)\,\epsilon_r' \qquad \ldots(5a)$$

and

$$\frac{\psi}{D} = -\frac{M_{r\theta}}{D} - (1-\mu)\,\epsilon_\theta' \qquad \ldots(5b)$$

where the prime (′) denotes differentiation with respect to r. From (2a)

$$\Delta\left(r\,\epsilon_r + r\,\frac{\partial w}{\partial r}\right) = \frac{h^2}{6\eta^2 D(1-\mu)}\,\Delta\,(r\,Q_r).$$

Using the operator relation

$$\Delta\left(r\,\frac{\partial}{\partial r}\right) = \left(r\,\frac{\partial}{\partial r}\right)\Delta + 2\Delta$$

to replace the term $\Delta\left(r\,\dfrac{\partial w}{\partial r}\right)$ and the relation (2 f) to replace $\Delta\,(r\,Q_r)$ we get

$$\Delta\,(r\,\epsilon_r) - \frac{1+\mu}{1-\mu}\,r\,\frac{\partial p}{\partial r} + 2p - \frac{12\eta^2}{h^2}\left(r\,\epsilon_r + r\,\frac{\partial w}{\partial r}\right) = 0. \qquad \ldots(6)$$

From (2g)

$$r\,Q_\theta = -D\,\frac{\partial p}{\partial \theta} - r\,\frac{\partial \psi}{\partial r}$$

and remembering that $\Delta\left(\dfrac{\partial p}{\partial \theta}\right) = 0$ we have

$$\Delta\,(r\,Q_\theta) + \frac{12\eta^2}{h^2}\left(r\,\frac{\partial \psi}{\partial r} + 2\psi\right) = 0. \qquad \ldots(7)$$

From (2b)

$$\Delta\,(r\,\epsilon_\theta) = \Delta\left[-\frac{\partial w}{\partial \theta} + \frac{h^2}{6\eta^2 D(1-\mu)}\,r\,Q_\theta\right]$$

$$= -\frac{\partial p}{\partial \theta} + \frac{h^2}{6\eta^2 D(1-\mu)}\,\Delta\,(r\,Q_\theta).$$

Using (2g) and (7) we get

$$\Delta\,(r\,\epsilon_\theta) = \frac{r}{D}\left(Q_\theta + \frac{\partial\psi}{\partial r}\right) - \frac{h^2}{6\eta^2\,D\,(1-\mu)}$$

$$\frac{12\eta^2}{h^2}\left(\cdot\ \frac{\partial\psi}{\partial r} + 2\psi\right)$$

i.e.

$$\Delta\,(r\,\epsilon_\theta) = r\,\frac{Q_\theta}{D} + \frac{1+\mu}{1-\mu}\,\frac{r}{D}\,\frac{\partial\psi}{\partial r} + \frac{4}{1-\mu}\,\frac{\psi}{D} = 0. \qquad \ldots(8)$$

Thus the differential equations (3), (4) and (6) are in the variables w, ϵ_r and M_θ. The equations (7), (8) and (1b) are in the variables Q_θ, ϵ_θ and $M_{r\theta}$.

We now seek the eigenfunction expansions of the functions w, p, ψ, ϵ_r, ϵ_θ, M_r, M_θ, $M_{r\theta}$, Q_r and Q_θ in the form

$$\left[w,\ p,\ \epsilon_r,\ \epsilon_\theta\right] = \sum_{(k)} a_k \exp(\lambda_k\,\theta)\left[w_k,\ p_k,\ \epsilon_{rk},\ \epsilon_{\theta k}\right] \qquad \ldots(9)$$

and

$$2\,\frac{1}{D}\left[\psi,\ M_r,\ M_\theta,\ M_{r\theta},\ Q_r,\ Q_\theta\right]$$

$$= \sum_{(k)} a_k \exp(\lambda_k\,\theta)\left[\psi_k,\ M_{rk},\ M_{\theta k},\ M_{r\theta k},\ Q_{rk},\ Q_{\theta k}\right] \qquad \ldots(10)$$

where λ_k is an eigenvalue.

Using the above expansions in (5a), (5b), (3), (4), (6), (7), (8) and (1b) we obtain

$$p_k = -\left[M_{\theta k} + (1-\mu)\,\epsilon'_{rk}\right] \qquad \ldots(11)$$

$$\psi_k = -\left[M_{r\theta k} + (1-\mu)\,\epsilon'_{\theta k}\right] \qquad \ldots(12)$$

$$r^2\,w''_k + r w'_k + \lambda_k^2\,w_k = r^2 p_k \qquad \ldots(13a)$$

$$r^2\,p''_k + r p'_k + \lambda_k^2\,p_k = 0 \qquad \ldots(13b)$$

$$r^2\,\epsilon''_{rk} + 3r\,\epsilon'_{rk} + \left(1 - \frac{12\eta^2\,r^2}{h^2}\right)\epsilon_{rk} - \frac{1+\mu}{1-\mu}\,r^2\,p'_k$$

$$+ 2r\,p_k - \frac{12\eta^2}{h^2}\,r^2\,w'_k + \lambda_k^2\,\epsilon_{rk} = 0 \qquad \ldots(13c)$$

$$r^2\, Q''_{\theta k} + 3r\, Q'_{\theta k} + Q_{\theta k} + \lambda_k^2\, Q_{\theta k} + \frac{12\eta^2}{h^2}$$

$$\left(r^2\, \psi'_k + 2r\, \psi_k \right) = 0 \qquad \text{...(14a)}$$

$$r^2\, \epsilon''_{\theta k} + 3r\, \epsilon'_{\theta k} + \epsilon_{\theta k} + \lambda_k^2\, \epsilon_{\theta k} - r^2\, Q_{\theta k}$$

$$+ \frac{1+\mu}{1-\mu}\, r^2\, \psi'_k + \frac{4}{1-\mu}\, r\, \psi_k = 0 \qquad \text{...(14b)}$$

$$r^2\, \psi''_k + r\, \psi'_k + \lambda_k^2\, \psi_k = \frac{12\eta^2}{h^2}\, r^2\, \psi_k. \qquad \text{...(14c)}$$

To pose (13) and (14) as system of adjoint differential equations we introduce

$$U_{1k} = w_k,\ U_{2k} = p_k = -\left[M_{\theta k} + (1-\mu)\, \epsilon'_{rk} \right],\ U_{3k} = \epsilon_{rk}$$

and

$$V_{1k} = Q_{\theta k},\ V_{2k} = \epsilon_{\theta k},\ V_{3k} = \psi_k = -\left[M_{r\theta k} + (1-\mu)\, \epsilon'_{\theta k} \right].$$

(13) may then be written as

$$r^2\, U''_{1k} + r\, U'_{1k} - r^2\, U_{2k} = -\lambda_k^2\, U_{1k} \qquad \text{...(15a)}$$

$$r^2\, U''_{2k} + r\, U'_{2k} = -\lambda_k^2\, U_{2k} \qquad \text{...(15b)}$$

$$r^2\, U''_{3k} + 3r\, U'_{3k} + \left(1 - 12\eta^2\, \frac{r^2}{h^2} \right) U_{3k}$$

$$- \frac{1+\mu}{1-\mu}\, r^2\, U'_{2k} + 2r\, U_{2k} - \frac{12\eta^2}{h^2}\, r^2\, U'_{1k} = -\lambda_k^2\, U^3{}_k \qquad \text{...(15c)}$$

(14) may be written as

$$r^2\, V''_{1k} + 3r\, V'_{1k}\, V_{1k} + \frac{12\eta^2}{h^2} \left(r^2\, V'_{3k} + 2r\, V_{3k} \right) = -\lambda_k^2\, V_{1k} \qquad \text{...(16a)}$$

$$r^2\, V''_{2k} + 3r\, V'_{2k} + V_{2k} - r^2\, V_{1k} + \frac{1+\mu}{1-\mu}\, r^2\, V'_{3k}$$

$$+ \frac{4}{1-\mu}\, r\, V_{3k} = -\lambda_k^2\, V_{2k} \qquad \text{...(16b)}$$

$$r^2 V''_{3k} + r\, V'_{3k} - \frac{12\eta^2}{h^2}\, r^2\, V_{3k} = -\lambda_k^2\, V_{3k}. \qquad \text{...(16c)}$$

The Orthogonality Relation

Equations (15) and (16) may be written in the vector form

$$L(\bar{U}_k) + \lambda_k^2 \ \bar{U}_k = 0 \qquad \text{...(17a)}$$

and

$$L^*(\bar{V}_k) + \lambda_k^2 \ \bar{V}_k = 0 \qquad \text{...(17b)}$$

where

$$\bar{U}_k = \begin{Bmatrix} U_{1k} \\ U_{2k} \\ U_{3k} \end{Bmatrix}, \ L(\bar{U}_k) = \begin{Bmatrix} r^2 U''_{1k} + r \ U'_{1k} - r^2 U_{2k} \\ r^2 U''_{2k} + r \ U'_{2k} \\ r^2 U''_{3k} + 3r \ U'_{3k} + \left(1 - \frac{12\eta^2}{h^2} r\right) U_{3k} \\ +2r \ U_{2k} - \frac{1+\mu}{1-\mu} r^2 U'_{2k} - \frac{12\eta^2}{h^2} r^2 U'_{1k} \end{Bmatrix}$$

and

$$\bar{V}_k = \begin{Bmatrix} V_{1k} \\ V_{2k} \\ V_{3k} \end{Bmatrix}, \ L^*(\bar{V}_k) = \begin{Bmatrix} r^2 V''_{1k} + r3 \ V'_{1k} + \frac{12\eta^2}{h^2} \left(r^2 V'_{3k} + 2r V_{3k}\right) \\ r^2 V''_{2k} + 3r \ V'_{2k} + V_{2k} - r^2 V_{1k} \\ + \left(\frac{1+\mu}{1-\mu}\right)\left(r^2 V'_{3k} + 2r \ V_{3k}\right) + 2r \ V_{3k} \\ r^2 V''_{3k} + r \ V'_{3k} - \frac{12\eta^2}{h^2} r^2 V_{3k} \end{Bmatrix}.$$

It follows from (17) that

$$\int_a^b [L(\bar{U}_k).\bar{V}_j - L^*(\bar{V}_j).\bar{U}_k] \, dr = \left(\lambda_j^2 - \lambda_k^2\right) \int_a^b (\bar{U}_k.\bar{V}_j) \, dr. \qquad \text{...(18)}$$

Expanding the integrand in full and carefully integrating by parts (18) reduces to

$$\left(\lambda_j^2 - \lambda_k^2\right) \int_a^b (w_k \, Q_{\theta j} - \epsilon_{\theta j} \, M_{\theta k} - \epsilon_{rk} \, M_{2\theta j}) \, dr$$

$$= \Big[r^2 \left(U'_{1k} \ V_{1j} - U_{1k} \ V'_{1j} + U'_{2k} \ V_{2j} - U_{2k} \ V'_{2j} + U'_{3k} V_{3j} - U_{3k} \ V'_{3j}\right)$$

$$+ r\,(U_{3k} \ V_{3j} - U_{2k} \ V_{2j} - U_{1k} \ V_{1j}) - \frac{1+\mu}{1-\mu} (r^2 \ U_{2k} \ V_{3j})$$

(*equation continued on p.* 439)

$$- \frac{12\eta^2}{h^2} (r^2 U_{1k} V_{3j}) \Big]_a^b + \left(\lambda_j^2 - \lambda_k^2 \right)(1-\mu) \Big[\epsilon_{\theta k}\, \epsilon_{\theta j} \Big]_a^b . \quad \ldots(19)$$

The components of the vectors $\bar{U}, \bar{V}, \bar{U}', \bar{V}'$ are related to the functions $w_k, \epsilon_{rk}, \epsilon_{\theta k}, M_{\theta k}, M_{r\theta k}, Q_{\theta k}$ and Q_{rk} as detailed in the Appendix (A.14 – A.25).

Upon considerable simplification of the right-hand side of (19) making use of the above relations we obtain ultimately

$$\left(\lambda_j^2 - \lambda_k^2 \right) \int_a^b (w_k\, Q_{\theta j} - \epsilon_{\theta j}\, M_{\theta k} - \epsilon_{rk}\, M_{r\theta j})\, dr$$

$$= r\,\lambda_j\,(w_j\, Q_{rk} - w_k\, Q_{rj}) + r\,\lambda_k\,(\epsilon_{\theta j}\, M_{r\theta k} - \epsilon_{\theta k}\, M_{r\theta j})$$

$$+ r\,\lambda_j\,(\epsilon_{rj}\, M_{rk} - \epsilon_{rk}\, M_{rj}) \Big]_a^b . \quad \ldots(20)$$

The following combinations of homogeneous boundary conditions may be prescribed on the curved boundaries of the plate :

Case (i) : Clamped : $w = \epsilon_\theta = \epsilon_r = 0$

Case (ii) : Simply supported : $w = M_r = M_{r\theta} = 0$

Case (iii) : Free : $Q_r = M_r = M_{r\theta} = 0$

Case (iv) : Elastically clamped : $w = \epsilon_\theta = 0$, $M_r = \alpha\epsilon_r$ where α is the moment required to produce a unit rotation ϵ_r.

Case (v) : Elastically supported : $M_r = M_{r\theta} = 0$, $Q_r = \beta w$ where β is the shear force Q_r required to produce a unit deflection w.

The right-hand side of (20) vanishes for all these conditions giving the desired orthogonality relation

$$\int_a^b (w_k\, Q_{\theta j} - \epsilon_{\theta j}\, M_{\theta k} - \epsilon_{rk}\, M_{r\theta j})\, dr = 0 \text{ for all } j \neq k. \quad \ldots(21)$$

Interchanging j and k above and adding the resulting equation to (21) we get the generalized orthogonality relation

$$\int_a^b (w_k\, Q_{\theta j} - \epsilon_{\theta j}\, M_{\theta k} - \epsilon_{rk}\, M_{r\theta j}$$

$$+ w_j\, Q_{\theta k} - \epsilon_{\theta k}\, M_{\theta j} - \epsilon_{rj}\, M_{r\theta k})\, dr = 0 \quad \ldots(22)$$

valid for all $j \neq k$.

The application of these orthogonality relations to boundary value problems in plate theory will form the basis of subsequent work by the author.

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Appendix

In deriving the results below expansions (9) and (10) have been used wherever necessary

Using (2a) and (2d) and remembering that

$$\frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} = 0$$

we get

$$p = -\frac{M_\theta}{D} - (1-\mu)\,\epsilon_r'' \qquad \text{...(A.1)}$$

Similarly using (2f) and (2g) we get

$$\frac{\partial Q_\theta}{\partial r} - \frac{1}{r}\frac{\partial Q_r}{\partial \theta} + \frac{Q_\theta}{r} = -\Delta\psi$$

$$= -12\,\frac{\eta^2}{h^2}\,\psi \text{ using (1b).}$$

From (2b), (2e) and above relation we get

$$\frac{\psi}{D} = -\frac{M_{r\theta}}{D} - (1-\mu)\,\epsilon_\theta' . \qquad \text{...(A.2)}$$

Using (2a), (2b) and (2e) we have

$$\frac{M_{r\theta}}{D} = -\frac{1-\mu}{2}\left(\frac{1}{r}\frac{\partial \epsilon_r}{\partial \theta} - \frac{\epsilon_\theta}{r} + \epsilon_\theta'\right). \qquad \text{...(A.3)}$$

From the above relation it follows that

$$\epsilon'_\theta = -\frac{2}{1-\mu}\frac{M_{r\theta}}{D} + \frac{1}{r}\left(\epsilon_\theta - \frac{\partial\epsilon_r}{\partial\theta}\right). \qquad ...(A.4)$$

Eliminating ϵ'_θ between (A.2) and (A.4) we get

$$\frac{\psi}{D} = \frac{M_{r\theta}}{D} + \frac{(1-\mu)}{r}\left(\frac{\partial\epsilon_r}{\partial\theta} - \epsilon_\theta\right). \qquad ...(A.5)$$

From (2a), (2b) and (2d) we get

$$\frac{M_\theta}{D} + \frac{\epsilon_r}{r} + \frac{1}{r}\frac{\partial\epsilon_\theta}{\partial\theta} + \mu\frac{\partial\epsilon_r}{\partial r}. \qquad ...(A.6)$$

Using (2c), (2d) and (3) we get

$$\frac{M_r}{D} + \frac{M_\theta}{D} = -(1+\mu)\,p. \qquad ...(A.7)$$

Eliminating p between (A.1) and the above relation we get

$$\mu\frac{M_\theta}{D} = \frac{M_r}{D} - (1-\mu^2)\,\epsilon'_r. \qquad ...(A.8)$$

Eliminating M_θ between (A.6) and the above relation we get

$$\epsilon'_r = \frac{M_r}{D} - \frac{\mu}{r}\left(\epsilon_r + \frac{\partial\epsilon_\theta}{\partial\theta}\right). \qquad ...(A.9)$$

From (A.8) and (A.9) we get

$$\frac{M_\theta}{D} = \mu\frac{M_r}{D} + (1-\mu^2)\left(\epsilon_r + \frac{\partial\epsilon_\theta}{\partial\theta}\right). \qquad ...(A.10)$$

From (A.7) and the above relation we get

$$p = -\frac{M_r}{D} - (1-\mu)\left(\epsilon_r + \frac{\partial\epsilon_\theta}{\partial\theta}\right). \qquad ...(A.11)$$

Using (2f) and remembering that $p = \Delta w$ we get

$$p' = -\frac{1}{D}\left(Q_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\right). \qquad ...(A.12)$$

Eliminating Q_θ between (2g) and (2b) we get

$$\epsilon_\theta + \frac{1}{r}\frac{\partial w}{\partial\theta} = -\frac{h^2}{6\eta^2(1-\mu)}\left(\frac{1}{r}\frac{\partial p}{\partial\theta} + \frac{\partial\psi}{\partial r}\right)$$

i.e.

$$\psi' = -\frac{6\eta^2(1-\mu)}{h^2}\left(\epsilon_\theta + \frac{1}{r}\frac{\partial w}{\partial\theta}\right) - \frac{1}{r}\frac{\partial p}{\partial\theta}. \qquad ..(A.13)$$

Using the definitions of the components of the vectors $\bar{U}$ and $\bar{V}$ we get

$$U_{1k} = w_k \quad \text{...(A.14)}$$

$$U_{2k} = p_k = - M_{rk} - (1 - \mu)(\epsilon_{rk} + \lambda_k \epsilon_{\theta k}) \text{ using (A.11)} \quad \text{...(A.15)}$$

$$U_{3k} = \epsilon_{rk} \quad \text{...(A.16)}$$

$$V_{1k} = Q_{\theta k} = \frac{6\eta^2 (1 - \mu)}{h^2}\left(\epsilon_{\theta k} + \frac{\lambda_k}{r} w_k \right) \text{ using (2b)} \quad \text{...(A.17)}$$

$$V_{2k} = \epsilon_{\theta k} \quad \text{...(A.18)}$$

$$V_{3k} = \psi_k = M_{r\theta k} + \frac{1 - \mu}{r}(\lambda_k \epsilon_{rk} - \epsilon_{\theta k}) \text{ using (A.5).} \quad \text{...(A.19)}$$

Using (2a) we get

$$U'_{1\theta k} = w'_{\theta k} = - \epsilon_{rk} + \frac{h^2}{6\eta^2 (1 - \mu)} Q_{rk}. \quad \text{...(A.20)}$$

Using (A.12) and (A.5) we get

$$U'_{2k} = w'_k = - Q_{rk} - \frac{\lambda_k M_{r\theta k}}{r} - \frac{(1 - \mu)\lambda_k}{r^2} (\lambda_k \epsilon_{rk} - \epsilon_{\theta k}). \quad \text{...(A.21)}$$

From (A.9) we have

$$U'_{3k} = \epsilon'_{rk} = M_{rk} - \frac{\mu}{r}(\epsilon_{rk} + \lambda_k \epsilon_{\theta k}). \quad \text{...(A.22)}$$

Using (A.17), (A.4) and (A.20) we get

$$V'_{1k} = Q'_{\theta k} = \frac{\lambda_k}{r} Q_{rk} + \frac{6\eta^2 (1 - \mu)}{h^2}\left[\frac{2}{1 - \mu} M_{r\theta k} + \frac{\epsilon_{\theta k}}{r} - \frac{2\lambda_k \epsilon_{rk}}{r}\right]. \quad \text{...(A.23)}$$

Using (A.4)

$$V'_{2k} = \epsilon'_{\theta k} = - \frac{2}{1 - \mu} M_{r\theta k} + \frac{1}{r}(\epsilon_{\theta k} - \lambda_k \epsilon_{rk}). \quad \text{...(A.24)}$$

From (A.13) and (A.11) we get

$$V'_{3k} = \psi'_k = - \frac{6\eta^2 (1 - \mu)}{h^2}\left(\epsilon_{\theta k} + \frac{\lambda_k}{r} w_k\right) + \left[M_{rk} + (1 - \mu)(\epsilon_{rk} + \lambda_k \epsilon_{\theta k})\right]. \quad \text{...(A.25)}$$

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A NOTE ON THE CHANDRASEKHAR'S *X*-FUNCTION

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A differentiability property of the Chandrasekhar's *X*-Function is analyzed.

Introduction

Chandrasekhar's X and Y Function play a vital role in the theory of radiative transfer[1] and neutron transport theory[2]. These functions are defined as follows :

$$\left.\begin{aligned} X(z) &= \lim_{x \to 0^+} J(x, z) \\ \text{and} \quad Y(z) &= \lim_{x \to t^-} J(x, z) \end{aligned}\right\} \qquad \ldots(1.1)$$

where $J(x, z)$ is the unique solution of the integral equation

$$J(x, z) = \exp(ixz) + \int_0^t K(x-y)\, J(y, z)\, dy. \qquad \ldots(1.2)$$

We denote the functions $X(z)$, $Y(z)$ and $J(x, z)$ by $X(z, t)$, $Y(z, t)$ and $J(x, z, t)$, respectively, to emphasize their dependence on t. In this note we analyse differentiability property of the X-function with respect to t.

Let T_t be an integral operator from $L_2[0, t]$ into itself defined as

$$T_t(f) = \int_0^t K(x-y)\, f(y)\, dy.$$

The adjoint operator T_t^* defined by

$$T_t^*(f) = \int_0^t \overline{K(y-x)}\, f(y)\, d(y).$$

Assuming that T_t is real and symmetric, i. e.

$$K(x) \text{ is real and } K(x) = K(-x) \qquad \ldots(1.3)$$

$$K(x) \text{ is locally square integrable} \qquad \ldots(1.4)$$

$$K(x) \in L_1(-\infty, \infty) \qquad \ldots(1.5)$$

and

$$1 - \hat{K}(\omega) \neq 0, \quad -\infty < \omega < \infty \qquad \ldots(1.6)$$

where $\hat{K}(\omega)$ denotes the Fourier transform of $K(x)$. Vittal Rao[3] proved that

$$\frac{d}{dt} X(0, t) = \alpha(t, t)\, X(0, t) \qquad \ldots(1.7)$$

where $\alpha(x, t)$ is the unique solution of the equation

$$\alpha(x, t) = K(x) + \int_0^t K(x - y)\, \alpha(y, t)\, dy. \qquad \ldots(1.8)$$

In this note we generalize this result for a larger class of operators, which satisfy, instead of (1.3) and (1.6) the following conditions

$$T_t \text{ is normal for all } t \qquad \ldots(1.3)'$$

$$1 - |\hat{K}(\omega)| \neq 0 \quad -\infty < \omega < \infty. \qquad \ldots(1.6)'$$

The generalization is established by the following theorem.

Theorem 1—The unique solution $\alpha(x, t)$ of (1.8) is differentiable with respect to t for almost every t and its derivative is $\alpha(t, t)\, \alpha_1(x = t, t)$ in the following sense.

$$\underset{h - 0}{Lt} \int_0^t \left| \frac{\alpha(x, t + h) - \alpha(x, t)}{h} - \alpha(t, t)\, \alpha_1(x - t, t) \right| dx = 0 \qquad \ldots(1.9)$$

where $\alpha_1(x, t)$ is the unique solution of the integral equation

$$\alpha_1(x, t) = K(x) + \int_{-t}^{0} K(x - y)\, \alpha_1(y, t)\, dy. \qquad \ldots(1.10)$$

Theorem 2—For real z

$$\frac{\partial}{\partial t} X(z, t) = \exp(itz)\, \overline{\alpha^*(t, t)}\, X^*(z, t) \qquad \ldots(1.11)$$

$$\frac{\partial}{\partial t} X^*(z, t) = \exp(-itz)\, \overline{\alpha(t, t)}\, X(z, t) \qquad \ldots(1.12)$$

where $X^*(z, t) = \underset{x - 0^+}{Lt} J^*(x, z, t)$, $J^*(x, z, t)$ is the unique solution of the equation

$$J^*(x, z, t) = \exp(-ixz) + \int_0^t K(y - x)\, J^*(y, z, t)\, dy \qquad \ldots(1.13)$$

and $\alpha^*(x, t)$ is the adjoint of $\alpha^*(x, t)$ given by

$$\alpha^*(x, y) = \overline{K(-x)} + \int_0^t \overline{K(y-x)}\, \alpha^*(y, t)\, dy. \qquad \ldots(1.14)$$

Proof of the Theorems

First we note that 1 is not an eigenvalue of the operator T and hence $(1 - T_t)^{-1}$ exists.

Proof of the Theorem 1—From (1.10) we get

$$\alpha_1(x-t, t) = K(x-t) + \int_{-t}^{0} K(x-t-y)\, \alpha_1(y, t)\, dy$$

$$= K(x-t) + \int_0^t K(x-s)\, \alpha_1(s-t, t)\, ds. \qquad \ldots(2.1)$$

Hence

$$\alpha_1(x-t, t) = (1 - T_t)^{-1}(K(x-t)). \qquad \ldots(2.2)$$

Now

$$\frac{\alpha(x, t) - \alpha(x, t-h)}{h} = \int_0^{t-h} K(x-y)\, \frac{\alpha(y, t) - \alpha(y, t-h)}{h}$$

$$+ \frac{1}{h} \int_{t-h}^{t} K(x-y)\, \alpha(y, t)\, dy.$$

Hence

$$\frac{\alpha(x, t) - \alpha(x, t-h)}{h} = (1 - T_{t-h})^{-1} \left[\frac{1}{h} \int_{t-h}^{t} K(x-y)\, \alpha(y, t)\, dy \right]. \qquad \ldots(2.3)$$

Define

$$D_h(x) = \frac{\alpha(x, t) - \alpha(x, t-h)}{h} - \alpha(t, t)\, \alpha_1(x-t, t).$$

Using (2.2) and (2.3) it can be easily verified as in Theorem 1 of Vittal Rao[3] that

$$\underset{h \to 0}{Lt} \left\| D_h(x) \right\|_1 = \underset{h \to 0}{Lt} \int_0^t \left| \frac{\alpha(x, t) - \alpha(x, t-h)}{h} - \alpha(t, t)\, \alpha_1(x-t, t) \right| dx = 0.$$

Similarly (1.9) can also be proved. This completes the proof.

Proof of the Theorem 2—Let R_t^* be the resolvent of the operator T_t^*. Then

$$R_t^*(x, y) - \overline{K(y-x)} = \int_0^t R_t^*(x, s)\, \overline{K(y-s)}\, ds. \quad \ldots(2.4)$$

From (2.4) an (1.8) it follows that

$$R_t^*(0, y) = \overline{\alpha(y, t)}\ . \quad \ldots(2.5)$$

The solution $J^*(x, z, t)$ of eqn. (1.13) is given by

$$J^*(x, z, t) = \exp(-ixz) + \int_0^t R_t^*(x, y) \exp(-iyz)\, dy. \quad \ldots(2.6)$$

Therefore

$$X^*(z, t) = \lim_{x \to 0^+} J^*(x, z, t)$$

$$= 1 + \int_0^t \overline{\alpha(y, t)} \exp(-iyz)\, dy. \quad \text{(From (2.5))}$$

Similarly

$$X(z, t) = 1 + \int_0^t \alpha^*(y, t) \exp(iyz)\, dy. \quad \ldots(2.7)$$

Now

$$\frac{X^*(z, t) - X^*(z, t-h)}{h} = \int_0^{t-h} \frac{\overline{\alpha(y, t)} - \overline{\alpha(y, t-h)}}{h}$$

$$\exp(-iyz)\, dy + \frac{1}{h} \int_{t-h}^{t} \overline{\alpha(y, t)} \exp(-iyz)\, dy. \quad \ldots(2.8)$$

From Theorem 1 it follows that for real z the first term in the right-hand side of (2.8) converges to $\overline{\alpha(t, t)} \int_0^t \overline{\alpha_1(y-t, t)} \exp(-iyz)\, dy$ and the second term converges to $\overline{\alpha(t, t)} \exp(-itz)$ as $h \to 0$.

Hence

$$\frac{\partial}{\partial t} X^* (z, t) = \overline{\alpha (t, t)} \left[\exp(-itz) + \int_0^t \overline{\alpha_1 (y - t, t)} \exp(-iyz)\, dy \right].$$

Substituting $y = t - x$ and observing that

$$\overline{\alpha_1 (-x, t)} = \alpha^* (x, t)$$

we get

$$\frac{\partial}{\partial t} X^* (z, t) = \overline{\alpha (t, t)} \exp(-itz) \left[1 + \int_0^t \alpha^* (x, t) \exp(ixz)\, dx \right].$$

This gives the formula (1.12). Similarly (1.11) can be proved.

A relation between the X-function and the eigenvalues had been obtained for symmetric integral operators[3]. Many spectral properties of symmetric operators had been extended for normal operators by Vittal Rao and and Sukavanam[4]. The results of this paper may be of use in obtaining a relation between eigenvalues and the X-function for normal operators.

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BESSEL TYPE FUNCTIONS INVOLVING DIRICHLET SERIES

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In this paper we construct an ordinary linear differential equation in the complex plane whose solutions in terms of Dirichlet series, found by the method developed in Das and Lahiri [*Rend, del Circolo Mat. di Palermo*, 33 (1984), 425-35], appear to satisfy certain relations analogous to the Bessel functions.

1. INTRODUCTION

In an earlier paper[3] we considered a class of ordinary linear differential equations in the complex plane for which we obtained solutions by a process in terms of Dirichlet series convergent in a half plane. For the analysis of that process we used arguments by Cesari[1,2]. In the present paper we take into consideration a particular second order ordinary linear differential equation (4) in the complex plane for which first we exhibit the solution in terms of Dirichlet series which results from the process by Das and Lahiri[3]. The same equation (4) could also be solved by power series and the corresponding solutions would have certain interesting properties. Furthermore the solutions of (4) in terms of Dirichlet series could be converted to power series by suitable transformations. But our main interest in the present paper is to formulate, independently of the power series, some basic properties of Dirichlet series solutions of (4) and to point out that these properties are similar to the classical properties enjoyed by Bessel functions. The present paper may be considered as a direct application of the theory initiated in Das and Lahiri[3].

Consider an ordinary linear differential equation in the complex plane

$$\frac{d^n w}{dz^n} + p_1(z)\,\frac{d^{n-1} w}{dz^{n-1}} + \dots + p_n(z)\, w = 0 \qquad \dots(1)$$

where $p_i(z)$ possesses, for each i, an absolutely convergent Dirichlet series (ordinary) expansion

$$p_i(z) = \sum_{k=1}^{\infty} p_{ik}/k^z$$

for $Rlz > \delta$ (say) and $i = 1, 2, \dots, n$.

The transformation

$$y_1 = w,\ y_2 = \frac{dw}{dz},\ \ldots,\ y_n = \frac{d^{n-1}w}{dz^{n-1}}$$

reduces (1) to the system

$$\frac{dy}{dz} = A(z)\,y \qquad \ldots(2)$$

where $y = (y_1, y_2, \ldots, y_n)^T$ and

$$A(z) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ -p_n & -p_{n-1} & -p_{n-2} & \ldots & -p_1 \end{bmatrix} = \sum_{k=1}^{\infty} A_k k^{-z}, \text{ say.}$$

In an earlier paper[3] we solved the system (2) in terms of Dirichlet series with a general $n \times n$ matrix $A(z)$ where we supposed that the entries $a_{ij}(z)$, $i, j = 1, 2, \ldots, n$, of the matrix $A(z)$ possess Dirichlet series expansions each absolutely convergent for $Rl z \geqslant \delta$.

If S denotes the set of all vector functions $w(z) = (w_1, w_2, \ldots, w_n)^T$

where

$$w_l(z) = \sum_{k=1}^{\infty} c_{kl}/k^z$$

is absolutely convergent for $Rl z \geqslant \delta$,

$$w(z) = \sum_{k=1}^{\infty} c_k/k^z,\ c_k = (c_{k1}, c_{k2}, \ldots, c_{kn})^T$$

then it has been shown in Das and Lahiri[3] that S is a Banach space with

$$\|w\| = \sum_{k=1}^{\infty} |c_k|/k^{\delta} = \sum_{k=1}^{\infty} \Big(\sum_{j=1}^{n} |c_{kj}|\Big)/k^{\delta}.$$

The following results are obtained in Das and Lahiri[3].

Lemma A—For any scalar λ, there is $w(z) = \sum_{k=1}^{\infty} c_k/k^z$ in S such that $y = w\, e^{\lambda z}$

satisfies the system

$$\Big(I \frac{d}{dz} - A\Big) y = (\lambda I - A_1)\, c_1\, e^{\lambda z}$$

where I is the $n \times n$ identity matrix.

Theorem A—For every eigenvalue λ of A_1 the system (2) possesses a solution.

The process involves the recursive relations for the determination of the c_k's as follows :

$$(\lambda I - A_1)\, c_1 = 0;$$

$$[A_1 - (\lambda + \log \tfrac{1}{2})\, I]\, c_2 = -\, A_2\, c_1;$$

$$[A_1 - (\lambda + \log \tfrac{1}{3})\, I]\, c_3 = -\, A_3\, c_1;$$

$$[A_1 - (\lambda + \log \tfrac{1}{4})\, I]\, c_4 = -\, A_4\, c_1 - A_2\, c_2;$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$[A_1 - (\lambda + \log \frac{1}{n})\, I]\, c_n = - \sum_{r_n, s_n} A_{r_n}\, c_{s_n} \qquad \ldots(3)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

where the summation on the right of the nth relation is extended over all r_n, s_n such that r_n, s_n are factors of n and $r_n . s_n = n$, $(r_k \neq 1)$, $r_n, s_n\ n = 2, 3, \ldots$.

In each of the cases when the eigen values

(i) do not differ by zero or by a logarithm of a positive integer;

(ii) do not differ by a logarithm of an integer (> 1);

(iii) differ by a logarithm of a positive integer (> 1),

we obtain in Das and Lahiri[3] two independent solutions of (1).

It was observed in Das and Lahiri[3] that if

$$y = (y_1, y_2, \ldots, y_n)^T = e^{\lambda z}\, (w_1, w_2, \ldots, w_n)^T$$

is a solution of (2) then $y_1 = e^{\lambda z}\, w_1\,(z)$ is a solution of (1).

2. Differential Equation and Solutions

Consider the following ordinary linear differential equation in the complex plane

$$\frac{d^2 w}{dz^2} + \frac{dw}{dz} + \{\alpha^2\, e^{\alpha z} - \tfrac{1}{4}\,(n^2 \alpha^2 - 1)\}\, w = 0 \qquad \ldots(4)$$

where $\alpha = \log \frac{1}{2}$ and n is an arbitrary constant.

The transformation $y_1 = w$, $y_2 = \dfrac{dw}{dz}$ reduces (4) to the system

$$\frac{dy}{dz} = A\,(z)\, y \qquad \ldots(5)$$

where $y = (y_1, y_2)^T$ and

$$A(z) = \begin{bmatrix} 0 & 1 \\ \frac{1}{4}(n^2\alpha^2 - 1) & -1 \end{bmatrix} 1^{-z} + \begin{bmatrix} 0 & 0 \\ -\alpha^2 & 0 \end{bmatrix} 2^{-z}$$

$$= A_1 .\ 1^{-z} + A_2 .\ 2^{-z}, \text{ say.}$$

Eigen values of A_1 are obtained from det $(A_1 - \lambda I) = 0$, i. e. from the equation

$$\lambda(\lambda + 1) - (\frac{n\alpha}{2} = \tfrac{1}{2})(\frac{n\alpha}{2} + \tfrac{1}{2}) - 0$$

which has the solutions

$$\lambda_1 = \frac{n\alpha}{2} - \tfrac{1}{2} \text{ and } \lambda_2 = -\frac{n\alpha}{2} - \tfrac{1}{2}.$$

If $y = (y_1, y_2)^T$

$$= e^{\lambda z}(w_1(z), w_2(z))^T$$

$$= e^{\lambda z}\left(\sum_{k=1}^{\infty} c_{k1}/kz, \sum_{k=1}^{\infty} c_{k2}/kz\right)^T$$

$$= e^{\lambda z} \sum_{k=1}^{\infty} c_k/kz$$

where $c_k = (c_{k1}, c_{k2})^T$, is a formal solution of (5) then we obtain using (3), the following relations

$$(\lambda I - A_1)\, c_1 = 0;$$

$$[A_1 - (\lambda + \log \tfrac{1}{2})\, I]\, c_2 = -A_2\, c_1;$$

$$[A_1 - (\lambda + \log \tfrac{1}{2}r) I]\, c_2 r = -A_2\, c_2 r - 1,\ r = 2, 3, \qquad \ldots(6)$$

since $A_k = 0$ for $k > 2$.

For any eigenvalue λ, we have

$$c_2 r_1 = \frac{-(\log \tfrac{1}{2})^2\, c_2 r - 1_1}{(\lambda + \log\tfrac{1}{2}r - \frac{n}{2}\log\tfrac{1}{2} + \tfrac{1}{2})(\lambda + \log\tfrac{1}{2}r + \frac{n}{2}\log\tfrac{1}{2} + \tfrac{1}{2})}$$

$$r = 1, 2, \ldots$$

$$= \frac{(-1)^r (\log \tfrac{1}{2})^{2r}\, c_{11}}{\prod_{p=1}^{r} (\lambda + p\log\tfrac{1}{2} - \frac{n}{2}\log\tfrac{1}{2} + \tfrac{1}{2})(\lambda + p\log\tfrac{1}{2} + \frac{n}{2}\log\tfrac{1}{2} + \tfrac{1}{2})}.$$

If $\lambda_1 = \dfrac{n\alpha}{2} - \tfrac{1}{2} = \dfrac{n}{2}\log\tfrac{1}{2} - \tfrac{1}{2}$ and n is not an integer, then the actual

computation yields

$$c_{21} = \frac{-c_{11}}{1.(n+1)},\ c_{41} = \frac{(-1)^2 c_{11}}{1.2.(n+1)(n+2)}$$

and in general

$$c_2r_1 = \frac{(-1)^r c_{11}}{1.2\ldots r\,(n+1)(n+2)\ldots(n+r)},\ r = 1, 2, \ldots.$$

From the first relation of (6), we have $c_1 = c_{11}(1,\lambda)^T$ and so taking $c_{11} = \dfrac{1}{\Gamma(n+1)}$, we obtain

$$y_1 = \exp\left(\frac{n}{2}\log\tfrac{1}{2} - \tfrac{1}{2}\right) z \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\,\Gamma(n+r+1)}(2^{-z})^r$$

$$= e^{-z/2}\, 2^{-nz/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\,\Gamma(n+r+1)}(2^{-z})^r$$

as a particular solution of (4) corresponding to the eigen value $\lambda_1 = \dfrac{n}{2}\log\tfrac{1}{2} - \tfrac{1}{2}$.

A second solution of (4) corresponding to the eigen value $\lambda_2 = -\dfrac{n}{2}\log\tfrac{1}{2} - \tfrac{1}{2}$ where n is not an integer, is given by

$$y_2 = e^{-zi2}\, 2^{nz/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1\ \Gamma(-n+r+1)}(2^{-z})^r.$$

If $n = \pm m$, where m is a positive integer, then the two eigen values of A_1 are $\lambda_1 = \dfrac{m}{2}\log\tfrac{1}{2} - \tfrac{1}{2}$ and $\lambda_2 = -\dfrac{m}{2}\log\tfrac{1}{2} - \tfrac{1}{2}$ so that $\lambda_1 - \lambda_1 = \log\tfrac{1}{2}{}_m$.

A particular solution of (4) corresponding to $\lambda_1 = \dfrac{m}{2}\log\tfrac{1}{2} - \tfrac{1}{2}$ is given by

$$y_1 = e^{-z/2}\, 2^{-mz/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\,\Gamma(m+r+1)}(2^{-z})^r. \qquad \ldots(7)$$

To obtain a solution of (4) corresponding to the eigen value $\lambda_2 = -\dfrac{m}{2}\log\tfrac{1}{2} - \tfrac{1}{2}$, we consider the formal series

$$y = \{(\lambda - \lambda_2)\, c_1 + \sum_{k=2}^{\infty} c_k l_k z\}\, e^{\lambda z}$$

satisfying the relation (cf. Lemma A)

$$(I\frac{d}{dz} - A)\, y = (\lambda I - A_1)\, c_1 (\lambda - \lambda_2)\, e^{\lambda z}.$$

The recursive relations in (6) yield $c_{21}, c_{2^2 1}, \ldots, c_{2^{m-1} 1}$ with $\lambda - \lambda_2$ as a factor. We recall that $c_r = 0$ if $r \neq 2^s$ for some positive integer s. Thus for $\lambda = \lambda_2$ we have $c_{21} = c_{2^2 1} = \ldots = c_{2^{m-1} 1} = 0$. In fact, for $1 \leqslant r \leqslant m$,

$$c_{2^r 1} = \frac{(-1)^r (\log \tfrac{1}{2})^{2r} (\lambda - \lambda_2)\, c_{11}}{\{(\prod_{p=1}^{r} (\lambda + p \log \tfrac{1}{2} - m/2 \log \tfrac{1}{2} + \tfrac{1}{2})\, (\lambda + p \log \tfrac{1}{2} + m/2 \log 1/2 + \tfrac{1}{2}\}} \qquad \ldots(8)$$

Since $\lambda + m \log \frac{1}{2} - \frac{m}{2} \log \frac{1}{2} + \frac{1}{2}$ is the only factor in the denominator of $c_{2^m 1}$ which vanishes for $\lambda = \lambda_2$, it follows that

$$c_{2^m 1} = \frac{(-1)^m (\log \tfrac{1}{2})^{2m}\, c_{11}}{\prod_{p=1}^{m-1} (\lambda + p \log \tfrac{1}{2} - m/2 \log \tfrac{1}{2} + \tfrac{1}{2}) \prod_{p=1}^{m} (\lambda + p \log \tfrac{1}{2} + m/2 \log \tfrac{1}{2} + \tfrac{1}{2})} \qquad \ldots(9)$$

is a rational function of λ in the neighbourhood of λ_2 and λ_2 is not a pole of $c_{2^m 1}$. Hence $c_{2^{m+r} 1}$, $r \geqslant 0$, can be calculated for $\lambda = \lambda_2$. In general, for $r \geqslant 1$

$$c_{2^{m+r} 1} = \frac{(-1)^{m+r} (\log \tfrac{1}{2})^{2(m+r)}\, c_{11}}{\prod_{p=1, p \neq m}^{m+r} (\lambda + p \log \tfrac{1}{2} - m/2 \log \tfrac{1}{2} + \tfrac{1}{2}) \prod_{p=1}^{m+r} (\lambda + p \log \tfrac{1}{2} + m/2 \log \tfrac{1}{2} + \tfrac{1}{2})}$$

$$= \frac{(-1)^r (\log \tfrac{1}{2})^{2r}\, c_{2^m 1}}{\prod_{p=1}^{r} \{\lambda + (m+p) \log \tfrac{1}{2} - m/2 \log \tfrac{1}{2} + \tfrac{1}{2}\} \{\lambda + (m+p) \log \tfrac{1}{2} + m/2 \log \tfrac{1}{2} + \tfrac{1}{2}\}} \cdot \qquad \ldots(10)$$

Choosing $c_{11} = (-1)^{m-1} (m-1)! (\log \frac{1}{2})^{-1}$, a particular solution of (4) corresponding to $\lambda_2 = -m/2 \log \frac{1}{2} - \frac{1}{2}$ is given by

$$y_2 = \exp(-m/2 \log \tfrac{1}{2} - \tfrac{1}{2})\, z \sum_{r=0}^{\infty} \frac{(-1)^{m+r}}{\Gamma(r+1)\, \Gamma(m+r+1)} (2^{-z})^{m+r}.$$

that is,

$$y_2 = (-1)^m e^{z/2}\, 2^{-mz/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\, \Gamma(m+r+1)} (2^{-z})^r \qquad \ldots(11)$$

Replacing r by $-m + r$, we obtain

$$y_2 = e^{-z/2}\, 2^{mz/2} \sum_{r=m}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\, \Gamma(-m+r+1)} (2^{-z})^r$$

(equation continued on p. 454)

$$= e^{-z/2}\, 2^{mz/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\, \bar{V}(-n+r+1)} (2^{-z})^r.$$

Thus we see that for any value of n

$$D_n(z) = e^{-z/2}\, 2^{-nz/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\, \Gamma(m+r+1)} (2^{-z})^r \qquad \dots(12)$$

and

$$D_{-n}(z) = e^{-z/2}\, 2^{nz/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\, \Gamma(-n+r+1)} (2^{-z})^r \qquad \dots(13)$$

are solutions of (4).

3. Theorems on Bessel Type Functions of the First Kind

If n is an integer, it is observed from (7) and (11) that $D_{-n}(z) = (-1)^n D_n(z)$ and so the solutions $D_n(z)$ and $D_{-n}(z)$ are not linearly indopendent. The following theorem strengthens the assertion.

Theorem 1—The Wronskian of $D_n(z)$ and $D_{-n}(z)$ is given by

$$W(D_n, D_{-n}) = \log \tfrac{1}{2}\, e^{-z} \; \frac{\sin n\pi}{\pi}.$$

Proof : If $P(z)$ and $Q(z)$ are two solutions of (4), it is not difficult to show that

$$W(P, Q) = A\, e^{-z},$$

where A is a constant.

The proof now follows from the fact that

$$D_n(z) = e^{-z/2}\, 2^{-nz/2} \frac{1}{\Gamma(n+1)} + O(2^{-z})$$

and

$$D_{-n}(z) = e^{-z/2}\, 2^{nz/2} \frac{1}{\Gamma(-n+1)} + O(2^{-z}).$$

Remark 1 : Since $W(D_n, D_{-n})$ vanishes only when n is an integer, it follows that $D_n(z)$ and $D_{-n}(z)$ are linearly independent solutions of (4) except when n is an integer.

In the case when $n = \pm m$, where m is a positive integer, a solution of (4) is given by

$$D_m(z) = e^{-z/2}\, 2^{-mz/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\,\Gamma(m+r+1)} \quad (2^{-z})^r.$$

To obtain a second independent solution of (4) we should apply Case III (p. 434) of Das and Lahiri[3]. A second independent solution of (4) can then be taken as $(-1)^m \dfrac{\partial y}{\partial \lambda}$ evaluated at $\lambda = \lambda_2$, where

$$y = (\lambda - \lambda_2)\, c_{11} + \sum_{r=1}^{\infty} c_2\, r_1\, (2^{-z})^r$$

and $c_2\, r_1$ are as in (8), (9) and (10). Hence a second independent solution of (4) is given by

$$\bar{D}_m(z) = z D_m(z) - e^{-z/2}\, 2^{mz/2} (\log \tfrac{1}{2})^{-1} \sum_{r=0}^{m-1} \frac{(m-r-1)}{r!} \quad (2^{-z})^r$$

$$- e^{-z/2}\, 2^{-mz/2} (\log \tfrac{1}{2})^{-1} \sum_{r=0}^{\infty} \frac{(-1)^r \{-H(m-1) + H(r) + (m+r)\}}{\Gamma(r+1)\;\Gamma(m+r+1)} \,(2^{-z})^r.$$

where $H(p) = 1 + \frac{1}{2} + \ldots + 1/p$, $H(0) = 0$.

If $n = 0$, then $\lambda_1 = \lambda_2 = \frac{1}{2}$. One solution of (4) is

$$D_0(z) = e^{-z/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \quad (2^{-z})^r$$

and the other independent solution, in view of Case II (p. 432) of Das and Lahiri[3], can be taken as

$$\bar{D}_0(z) = (1+z)\, D_0(z) - 2e^{-z/2} (\log \tfrac{1}{2})^{-1} \sum_{r=1}^{\infty} \frac{(-1)^r}{(r!)^2} \;\; H(r)\,(2^{-z})^r.$$

Theorem 2 (Recurrrence relations)—(i) $nD_n(z) = 2^{-z/2}\,[D_{n-1}(z) + D_{n+1}(z)]$;

(ii) $2D'_n(z) + D_n(z) = \log \frac{1}{2}\, 2^{-z/2}\,[D_{n-1}(z) - D_{n+1}(z)]$.

PROOF : Multiplying $D_n(z)$ by $2^{-nz/2}$ and differentiating with respect to z, we obtain

$$\frac{d}{dz}(2^{-nz/2} D_n(z)) = -\tfrac{1}{2}.\, 2^{-nz/2} D_n(z) + \log \tfrac{1}{2}\, e^{-z/2}$$

(equation continued on p. 456)

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\,\Gamma(n+r)} (2^{-z})^{r+n}$$

$$= -\tfrac{1}{2}\, 2^{-nz/2}\, D_n(z) + \log \tfrac{1}{2} \,.2^{-\frac{n+12}{2}}\, D_{n-1}(z).$$

This gives

$$D'_n\ (z) + \frac{n}{2} \log \tfrac{1}{2}\, D_n(z) = -\tfrac{1}{2}\, D_n(z) + \log \tfrac{1}{2}.\ 2^{-z/2}\, D_{n-1}(z).$$

that is,

$$D'_n\ (z) + \left(\frac{n}{2} \log \tfrac{1}{2} + \tfrac{1}{2}\right) D_n(z) - \log \tfrac{1}{2}.2^{-z/2}\, D_{n-1}(z) = 0. \qquad \ldots(14)$$

Again, multiplying $D_n(z)$ by $2^{nz/2}$ and differentiating with respect to z, we obtain

$$\frac{d}{dz}\,(2^{nz/2}\, D_n(z)) = -\tfrac{1}{2}\, 2^{nz/2}\, D_n(z) + \log \tfrac{1}{2}\, e^{-z/2}$$

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{\Gamma(r)\,\Gamma(n+r+1)} (2^{-z})^r$$

$$= -\tfrac{1}{2}\, 2^{nz/2}\, D_n(z) + \log \tfrac{1}{2}\, e^{-z/2}$$

$$\times \sum_{s=0}^{\infty} \frac{(-1)^{s+1}\,(2^{-z})^{s+1}}{\Gamma(s+1)\,\Gamma(n+1+s+1)} = -\tfrac{1}{2}^{nz/2}\, D_n(z) - \log \tfrac{1}{2}\, 2^{(n-1)z/2}$$

$$\times\ D_{n+1}(z).$$

This gives

$$D'_n(z) - n/2 \log \tfrac{1}{2}\, D_n(z) = -\tfrac{1}{2}\, D_n(z) - \log \tfrac{1}{2}.2^{-z/2}\, D_{n+1}(z),$$

that is,

$$D'_n\ (z) - (n/2 \log \tfrac{1}{2} - \tfrac{1}{2})\, D_n(z) + \log \tfrac{1}{2}\, 2^{-z/2}\, D_{n+1}(z) = 0. \qquad \ldots(15)$$

From (14) and (15) we obtain

$$nD_n(z) = 2^{-z/2}\,[D_{n-1}(z) + D_{n+1}(z)]$$

and

$$2D'_n\ (z) + D_n(z) = \log \tfrac{1}{2}.\ 2^{-z/2}\,[D_{n-1}(z) - D_{n+1}(z)].$$

This proves the theorem.

Theorem 3 (Generating function)—

$$\exp\{-z/2 + 2^{-z/2}(t - 1/t)\} = \sum_{n=-\infty}^{\infty} D_n(z).\, t^n,\ 0 < \delta \leqslant |t| < \infty.$$

PROOF : We have

$$\exp(2^{-z/2}\, t) = 1 + 2^{-z/2}\, t + \frac{(2^{-z/2})^2 t^2}{2!} + \ldots$$

and

$$\exp(-\, 2^{-z/2}/t) = 1 - \frac{(2^{-z/2})}{1!}\ \frac{1}{t} + \frac{(2^{-z/2})^2}{2!}\ \frac{1}{t^2} - \cdots .$$

Then

$$\exp\left\{2^{-z/2}\left(t - \frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} \frac{t^n (2^{-z/2})^n}{n!} \left\{1 - \frac{(2^{-z/2})^2}{n+1} + \frac{(2^{-z/2})^4}{2!\,(n+1)(n+2)} - \ldots\right\}.$$

So,

$$\exp\left\{-\frac{z}{2} + 2^{-z/2}\left(t - \frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} \left\{e^{-z/2}\, 2^{-nz/2} \times \sum_{r=0}^{\infty} \frac{(-1)^r (2^{-z})^r}{\Gamma(r+1)\,\Gamma(n+r+1)}\right\} t^n$$

$$= \sum_{n=-\infty}^{\infty} D_n(z)\, t^n.$$

This proves the theorem.

Theorem 4 (Integral representation)—For integral values of n

$$D_n(z) = \frac{1}{\pi} \int_0^{\pi} e^{-z/2} \cos(n\theta - 2^{(1-z)/2} \sin\theta)\, d\theta.$$

PROOF : In view of Theorem 3

$$D_n(z) = \frac{1}{2\pi i} \int_C e^{-z/2}\, e^{2^{-z/2}}\left(t - \frac{1}{t}\right) t^{-n-1}\, dt$$

where C is a closed contour enclosing the point $t = 0$. Taking C to be the unit circle $|t| = 1$, we obtain

$$D_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-z/2}\ e^{-in\theta}\ e^{2^{-z/2}\ 2i \sin\theta}\ d\theta$$

$$= \frac{1}{\pi} \int_{0}^{\pi} e^{-z/2} \cos\left(n\theta - 2^{(1-z)/2} \sin\theta\right) d\theta$$

the imaginary part vanishes since the integrand is an odd function. This proves the theorem.

4. Bessel Type Functions of The Second Kind

It is clear that the function $M_n(z)$ defined by

$$M_n(z) = \frac{D_n(z) \cos n\pi - D_{-n}(z)}{\sin n\pi} \qquad \ldots(16)$$

is a solution of (4) for non-integral values of n. For an integer m, we define

$$M_m(z) = \lim_{n \to m} M_n(z).$$

Applying L'Hospital's theorem, we obtain

$$M_m(z) = \frac{1}{\pi}\left\{ \frac{dD_n(z)}{dn}\Big|_{n=m} - (-1)^m \frac{dD_{-n}(z)}{dn}\Big|_{n=m} \right\}. \qquad \ldots(17)$$

The above relation further gives

$$M_{-m}(z) = (-1)^m\ M_m(z).$$

For integral values of n, it may be noted that $D_n(z)$ and $D_{-n}(z)$ converge uniformly in n and this gives that $M_n(z)$ coverges uniformly in n and consequently the function $M_n(z)$ is a solution (4) for any value of n. It is observed that this solution resembles the classical Bessel's function of the second kind.

Theorem 5 (cf. Theorem 2)—(i) $n\ M_n(z) = 2^{-z/2}\ [M_{n-1}(z) - M_{n+1}(z)]$;

(ii) $2\ M_n'(z) + M_n(z) = \log \frac{1}{2}.\ 2^{-z/2}\ [M_{n-1}(z) + M_{n+1}(z)]$.

Proof : Following the proof of Theorem 2, we obtain

$$\frac{d}{dz}\left(2^{-nz/2}\ D_{-n}(z)\right) = -\tfrac{1}{2}\ 2^{-nz/2}\ D_{-n}(z) - 2^{-(n+1)z/2} \log \tfrac{1}{2}\ D_{-n+1}(z);$$

$$\frac{d}{dz}\left(2^{nz/2}\ D_{-n}(z)\right) = \tfrac{1}{2}\ 2^{nz/2}\ D_{-n}(z) + 2^{(n-1)z/2} \log \tfrac{1}{2}\ D_{-n-1}(z).$$

Utilising (16) and simplifying, we obtain, for non-integer n,

$$\frac{d}{dz}\left(2^{-nz/2}\ M_n(z)\right) = -\tfrac{1}{2}\ 2^{-nz/2}\ M_n(z) + 2^{-(n+1)z/2} \log \tfrac{1}{2}\ M_{n-1}(z);$$

$$\frac{d}{dz}\,(2^{nz/2}\,M_n(z)) = -\tfrac{1}{2}\,2^{nz/2}\,M_n(z) - 2^{(n-1)z/2}\log\tfrac{1}{2}\,M_{n+1}(z).$$

The rest of the proof is analogous to that of Theorem 2. We omit the details. This proves the theorem.

We present now the series expansion of the function $M_m(z)$ for integral values of m (> 0).

Theorem 6—$$M_m(z) = -\frac{1}{\pi}\,e^{-z/2}\,2^{mz/2}\sum_{r=0}^{m-1}\frac{(m-r-1)!}{r!}\,(2^{-z})^r$$

$$-\frac{1}{\pi}\,e^{-z/2}\,2^{-mz/2}\sum_{r=0}^{\infty}\frac{(-1)^r\,(2^{-z})^r}{r!\,(m+r)!}\,\{z\log\tfrac{1}{2}-\psi(r+1)-\psi(m+r+1)\}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ and the first sum is equal to zero if $m = 0$.

Proof : Since the series (12) for $D_n(z)$ converge uniformly in n, we can differentiate term by term with respect to n and obtain

$$\left.\frac{dD_n(z)}{dn}\right|_{n=m} = \frac{z}{2}\log\tfrac{1}{2}\,e^{-z/2}\,2^{-mz/2}\sum_{r=0}^{\infty}\frac{(-1)^r\,(2^{-z})^r}{\Gamma(r+1)\,\Gamma(m+r+1)}$$

$$+\,e^{-z/2}\,2^{-mz/2}\sum_{r=0}^{\infty}\frac{(-1)^r\,(2^{-z})^r}{\Gamma(r+1)}\,\frac{d}{dm}\left\{\frac{1}{\Gamma(m+r+1)}\right\}$$

$$= \frac{z}{2}\log\tfrac{1}{2}\,e^{-z/2}\,2^{-mz/2}\sum_{r=0}^{\infty}\frac{(-1)^r\,(2^{-z})^r}{\Gamma(r+1)\,\Gamma(m+r+1)}$$

$$-\,e^{-z/2}\,2^{-mz/2}\sum_{r=0}^{\infty}\frac{(-1)^r\,(2^{-z})^r}{\Gamma(r+1)\,\Gamma(m+r+1)}$$

$$\times\left\{\gamma+\frac{1}{m+r+1}-\sum_{s=1}^{\infty}\frac{m+r+1}{s\,(m+r+1+s)}\right\}$$

$$= e^{-z/2}\,2^{-mz/2}\sum_{r=0}^{\infty}\frac{(-1)^r\,(2^{-z})^r}{\Gamma(r+1)\,\Gamma(m+r+1)}$$

$$\times\left\{\frac{z}{2}\log\tfrac{1}{2}-\psi(m+r+1)\right\},$$

where γ is the Euler's constant.

Similarly using (13), we obtain

$$\left.\frac{dD_{-n}(z)}{dn}\right|_{n=m} = e^{-z/2}\, 2^{nz/2} \sum_{r=0}^{\infty} \frac{(-1)^r\,(2^{-z})^r}{\Gamma(r+1)\,\Gamma(-n+r+1)}$$

$$\times \left.\left\{-\frac{z}{2}\log\tfrac{1}{2} + \psi(-n+r+1)\right\}\right|_{n=m} .$$

For $r = 0, 1, \ldots, m-1$, both $\Gamma(-n+r+1)$ and $\psi(-n+r+1)$ tend to ∞ as $n \to m$ so that the first m terms of the last series become indeterminate.

We note that

$$\frac{1}{\Gamma(-n+r+1)} = \frac{\Gamma(n-r)\sin\pi(n-r)}{\pi}$$

so that

$$\frac{\psi(-n+r+1)}{\Gamma(-n+r+1)} = \frac{\Gamma'(-n+r+1)}{\{\Gamma(-n+r+1)\}^2} = \Gamma(n-r)\sin\pi(n-r)$$

$$\times\frac{\psi(n-r) + \pi\cot\pi(n-r)}{\pi} .$$

Hence for $r = 0, 1, \ldots, m-1$

$$\lim_{n\to m}\frac{\psi(-n+r+1)}{\Gamma(-n+r+1)} = (-1)^{m-r}(m-r-1)! .$$

Thus

$$\left.\frac{dD_{-n}(z)}{dn}\right|_{n=m} = (-1)^m e^{-z/2}\, 2^{mz/2} \sum_{r=0}^{m-1} \frac{(m-r-1)!}{r!}(2^{-z})^r$$

$$+ e^{-z/2}\, 2^{mz/2} \sum_{r=n}^{\infty} \frac{(-1)^r\,(2^{-z})^r}{\Gamma(r+1)\,\Gamma(-m+r+1)}$$

$$\times\left\{-\frac{z}{2}\log\tfrac{1}{2} + \psi(-m+r+1)\right\}$$

$$= (-1)^m e^{-z/2}\, 2^{mz/2} \sum_{r=0}^{m-1} \frac{(m-r-1)!}{r!}(2^{-z})^r$$

$$+ (-1)^m e^{-z/2}\, 2^{mz/2} \sum_{r=0}^{\infty} \frac{(-1)^r\,(2^{-z})^{m+r}}{r!\,(m+r)!}$$

(*equation continued on p. 461*)

$$\times \left\{-\frac{z}{2}\log\tfrac{1}{2}+\psi(r+1)\right\}.$$

Hence from (17), we obtain, for integral values of m ($\geqslant 0$)

$$M_m(z) = -\frac{1}{\pi} e^{-z/2}\, 2^{mz/2} \sum_{r=0}^{m-1} \frac{(m-r-1)!}{r!} (2^{-z})^r$$

$$+\frac{1}{\pi} e^{-z/2}\, 2^{-mz/2} \sum_{r=0}^{\infty} \frac{(-1)^r (2^{-z})^r}{r!\,(m+r)!}$$

$$\times \left\{ z\log\tfrac{1}{2} - \psi(r+1) - \psi(m+r+1) \right\},$$

where the first sum should be set equal to zero if $m = 0$.

This proves the theorem.

Utilising relation (12), we obtain an equivalent statement of Theorem 6 as follows :

$$M_m(z) = \left(\frac{1}{\pi}\log\tfrac{1}{2}\right) z\, D_m(z) - \frac{1}{\pi} e^{-z/2}\, 2^{mz/2} \sum_{r=0}^{m-1} \frac{(m-r-1)!}{r!} (2^{-z})^r$$

$$-\frac{1}{\pi} e^{-z/2}\, 2^{-mz/2} \sum_{r=0}^{\infty} \frac{(-1)^r (2^{-z})^r}{r!\,(m+r)!} \{\psi(r+1)+\psi(m+r+1)\}.$$

5. Reciprocity Relation

Let $D_n(z)$ be defined by the formula

$$\exp\left(-\frac{z}{2}\right).\exp\left(2^{-z/2}\left(t-\frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} D_n(z) t^n,\ 0<\delta\leqslant |t|<\infty, \qquad \ldots(18)$$

and let us call the function on the left, the generating function for $D_n(z)$. By actual computations we see that

$$\sum_{n=-\infty}^{\infty} D_n(z)t^n = \sum_{n=-\infty}^{\infty} \left\{\exp\left(-\frac{z}{2}\right) 2^{-nz/2} \sum_{r=0}^{\infty} \frac{(-1)^r (2^{-z})^r}{\Gamma(r+1)\,\Gamma(n+r+1)}\right\} t^n$$

valid for all values of z and $0<\delta\leqslant |t|<\infty$. This gives the Dirichlet series expansion of $D_n(z)$ as

$$D_n(z) = \exp\left(-\frac{z}{2}\right) 2^{-nz/2} \sum_{r=0}^{\infty} \frac{(-1)^r (2^{-z})^r}{\Gamma(r+1)\,\Gamma(n+r+1)} . \qquad \ldots(19)$$

In view of Theorem 2, we have

$$nD_n(z) = 2^{-z/2}\,[D_{n-1}(z) + D_{n+1}(z)]; \qquad ..(20)$$

$$2\ D'_n(z) + D_n(z) = \log \tfrac{1}{2}\, 2^{-z/2}\,[D_{n-1}(z) - D_{n+1}(z)]. \qquad ..(21)$$

Eliminating $D_n(z)$ from (20) and (21), we obtain

$$D'_n(z) + \left(\frac{n}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\right) D_n(z) = \log\tfrac{1}{2}\, 2^{-z/2}\, D_{n-1}(z). \qquad ..(22)$$

Differentiating (22) with respect to z

$$D''_n(z) + \left(\frac{n}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\right) D'_n(z) = \tfrac{1}{2}(\log\tfrac{1}{2})^2\, 2^{-z/2}\, D_{n-1}(z)$$

$$+ \log\tfrac{1}{2}.\ 2^{-z/2}\, D'_{n-1}(z).$$

Substituting for $D'_{n-1}(z)$ from (22), (n replaced by $n-1$), we obtain

$$\begin{aligned} D''_n(z) + \left(\frac{n}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\right) D'_n(z) = {} & \tfrac{1}{2}(\log\tfrac{1}{2})^2\, 2^{-z/2}\, D_{n-1}(z) \\ & + (\log\tfrac{1}{2})^2\, 2^{-z}\, D_{n-2}(z) \\ & - \log\tfrac{1}{2}\, 2^{-z/2}\left(\frac{n-1}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\right) \\ & \times D_{n-1}(z). \end{aligned}$$

Eliminating $D_{n-2}(z)$ from the above relation and (22) with n replaced by $n-1$, we get

$$\begin{aligned} D''_n(z) + \left(\frac{n}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\right) D'_n(z) &= (n-1)(\log\tfrac{1}{2})^2\, 2^{-z/2}\, D_{n-1}(z) \\ &\quad - (\log\tfrac{1}{2})^2\, 2^{-z}\, D_n(z) - \log\tfrac{1}{2}\, 2^{-z/2}\left(\frac{n-2}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\right) D_{n-1}(z) \\ &= (n-1)\log\tfrac{1}{2}\left\{ D'_n(z) + \left(\frac{n}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\right) D_n(z)\right\} \\ &\quad - \left(\frac{n-2}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\right) D'_n(z) - \left(\frac{n}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\right) \\ &\quad \times \left(\frac{n}{2}\log\tfrac{1}{2} + \tfrac{1}{2} - \log\tfrac{1}{2}\right) D_n(z) - (\log\tfrac{1}{2})^2\, 2^{-z}\, D_n(z), \text{ using (22).} \end{aligned}$$

This gives

$$D_n''(z) + D_n'(z) + \left\{ (\log \tfrac{1}{2})^2 \, 2^{-z} - \left(\frac{n}{2} \log \tfrac{1}{2} + \tfrac{1}{2} \right) \left(\frac{n}{2} \log \tfrac{1}{2} - \tfrac{1}{2} \right\} \times D_n(z) = 0. \qquad \ldots(23)$$

This shows that $D_n(z)$ is a solution of the equation

$$\frac{d^2w}{dz^2} + \frac{dw}{dz} + \left\{ \alpha^2 e^{\alpha z} - \tfrac{1}{4} (n^2 \alpha^2 - 1) \right\} w = 0 \qquad \ldots(24)$$

where $\alpha = \log \frac{1}{2}$ and n is an arbitrary constant.

The differential equation (24) is eqn. (4) with which we started.

Remark 2 : If in the equation (4) we take $\alpha = \log 1/p$ where p is a positive integer (> 2) instead of $\log \frac{1}{2}$ then except longer calculations, analogous results would be obtained.

References

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GROWTH AND APPROXIMATION OF GENERALIZED BI-AXIALLY SYMMETRIC POTENTIALS

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The present paper deals with growth and approximation of solutions (not necessarily entire) of certain elliptic partial differential equations. These solutions are called Generalized Bi-axially Symmetric Potentials (GBSP's). The GBSP's are taken to be regular in a finite hyperball and influence of the growth of their maximum moduli on the rate of decay of their approximation errors in both sup norm and L^{δ}-norm, $1 \leqslant \delta < \infty$, is studied.

1. INTRODUCTION

Generalized bi-axially symmetric potentials (GBSP's) are the solutions of the elliptic partial differential equation

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{2\alpha+1}{y}\frac{\partial H}{\partial y} + \frac{2\beta+1}{x}\frac{\partial H}{\partial x} = 0,\ \alpha, \beta > -1/2. \quad \ldots(1.1)$$

which are even in x and y (Gilbert[4]). A polynomial of degree n which is even in x and y is said to be a GBSP polynomial of degree n if it satisfies (1.1). A GBSP H, regular about origin, can be expanded as

$$H \equiv H(r,\theta) = \sum_{n=0}^{\infty} a_n r^{2n} P_n^{(\alpha,\beta)}(\cos 2\theta) \quad . \quad \ldots(1.2)$$

where, $x = r\cos\theta$, $y = r\sin\theta$ and $P_n^{(\alpha,\beta)}(t)$ are Jacobi polynomials.

Let $D_R = \{(x,y) : x^2 + y^2 < R^2\}$, $0 < R \leqslant \infty$ and $\bar{D}_R$ be the closure of D_R. A GBSP H is said to be regular in D_R if the series (1.2) converges uniformly on compact subsets of D_R. Let H_R be the class of all GBSP's H regular in D_R, for every $R' \leqslant R$ but for no $R' > R$. The functions in the class H_∞ are called entire GBSP's.

Recently, McCoy[7,8] considered the approximation of an entire GBSP H by GBSP polynomials and found the rate of decay of approximation errors

$$E_n(H, 1) = \inf_{g\in\pi_n} \|H - g\|_1 = \inf_{g\in\pi_n} \{ \max_{(x,y)\in\bar{D}_1} |H(x,y)| \}$$

and

$$E_{n,\delta}(H, 1) = \inf_{g \in \pi_n} \|H - g\|_{1,\delta}$$

$$= \inf_{g \in \pi_n} \{\iint_{\bar{D}_1} \mu(x, y) \mid H(x, y) \mid^{\delta} dx\, dy\}^{1/\delta}$$

μ a weight function, $1 \leqslant \delta < \infty$, in terms of the growth parameters associated with the maximum modulus function

$$M(r, H) = \max_{\theta} \mid H(r, \theta) \mid .$$

The effect of rate of growth of $M(r, H)$ on the coefficients a_n in (1.2) of an entire GBSP H were studied by Fryant[3].

However, for a GBSP H which is not entire, no attempt seems to have been made so far to study the approximation errors or the coefficients in (1.2), vis-a-vis, the rate of growth of $M(r, H)$. The present paper is an effort in this direction.

A GBSP H is said to be regular on $\bar{D}_{R_0}$, $0 < R_0 < \infty$, the closure of D_{R_0}, if it is regular in $D_{R'}$ for some $R' > R_0$. We denote by $\bar{H}_{R_0}$ the class of all GBSP's H regular on $\bar{D}_{R_0}$. For $H \in \bar{H}_{R_0}$, set

$$\| H \|_{R_0} = \max_{(x, y) \in \bar{D}_{R_0}} \mid H(x, y) \mid$$

$$\| H \|^1_{R_0, \delta} = \left(\int_0^{2\pi} w(R_0, \theta) \mid H(R_0, \theta) \mid^{\delta} d\theta \right)^{1/\delta}, \; 1 \leqslant \delta < \infty \quad \ldots(1.3)$$

$$\| H \|^2_{R_0, \delta} = \left(\iint_{\bar{D}_{R_0}} \varpi(x, y) \mid H(x, y) \mid^{\delta} dx\, dy \right)^{1/\delta}, \; 1 \leqslant \delta < \infty \quad \ldots(1.4)$$

where the functions w and ϖ are positive and integrable (in the sense of Lebesgue) such that $1/w$ and $1/\varpi$ are bounded. Then $\| \cdot \|_{R_0}$ is the uniform norm on $\bar{H}_{R_0}$ whlie $\| \cdot \|^1_{R_0, \delta}$ and $\| \cdot \|^2_{R_0, \delta}$ are L^{δ}-norms on $\bar{H}_{R_0}$. For $H \in \bar{H}_{R_0}$, the approximation errors $E_n(H, R_0)$, $E^1_{n, \delta}(H, R_0)$ and $E^2_{n, \delta}(H, R_0)$ are defined as

$$E_n(H, R_0) = \inf_{g \in \pi_n} \| H - g \|_{R_0} \quad \ldots(1.5)$$

$$E^1_{n, \delta}(H, R_0) = \inf_{g \in \pi_n} \| H - g \|^1_{R_0, \delta} \quad \ldots(1.6)$$

$$E^2_{n,\delta}(H, R_0) = \inf_{g\in\pi_n} \| H - g \|^2_{R_0,\delta} \qquad ..(1.7)$$

where π_n consists of all GBSP polynomials of degree at most $2n$.

A GBSP $H \in H_R$, $0 < R < \infty$, is said to be of order ρ if

$$\rho \equiv \rho(H, R) = \limsup_{r\to R} \frac{\log^+ \log^+ M(r, H)}{\log (R/(R-r))}$$

where, $\log^+ x = \max(0, \log x)$. If $0 < \rho < \infty$, then the type T of H is defined as

$$T \equiv T(H, R) = \limsup_{r\to R} \frac{\log^+ M(r, H)}{(R/(R-r))^\rho}. \qquad ...(1.8)$$

In Section 2, we obtain characterizations of rate of decay of approximation error $E_n(H, R_0)$ in terms of order ρ and type T of $H \in H_R$, $0 < R_0 < R < \infty$. The Characterizations of rate of decrease of $E^i_{n,\delta}(H, R_0)$, $i = 1, 2$, and a_n in terms of these growth parameters of $H \in H_R$ are found in Section 3.

2. Approximation Error $E_n(H, R_0)$

We need the following lemmas :

Lemma 1—Let $H \in H_R$, $R > R_0$. Then, there exist GBSP polynomials $g_n \in \pi_n$ such that

$$\| H - g_n \|_{R_0} \leq KM(r, H)(n+1)^{q+1/2}(R_0/r)^{2(n+1)}$$

for all r sufficiently near to R and all sufficiently large values of n. Here K is a constant independent of n and r and $q = \max(\alpha, \beta)$.

Proof : Set

$$g_n = \sum_{k=0}^{n} a_k r^{2k} P_k^{(\alpha,\beta)}(\cos 2\theta).$$

Then $g_n \in \pi_n$. Using (1.2), we get

$$\| H - g_n \|_{R_0} \leq \sum_{k=n+1}^{\infty} | a_k | R_0^{2k} | P_k^{(\alpha,\beta)}(\cos 2\theta) |$$

$$\leq \frac{1}{\Gamma(q+1)} \sum_{k=n+1}^{\infty} | a_k | R_0^{2k} \Gamma(k+q+1)/\Gamma(k+1) \qquad ...(2.1)$$

since, we have[9] (p. 168)

$$\max_{-1\leqslant t\leqslant 1} \mid P_k^{(\alpha,\beta)}(t) \mid = \frac{\Gamma(k+n+1)}{\Gamma(q+1)\,\Gamma(k+1)}, \quad q = \max(\alpha,\beta). \qquad \ldots(2.2)$$

For $H \in H_R$, we have ([3, (8)])

$$\mid a_k \mid \leqslant \frac{M(r, H)}{r^{2k}} \left[(2k + \alpha + \beta + 1)\, A(k, \alpha, \beta)\, A(\alpha, \beta)\right]^{1/2} \qquad \ldots(2.3)$$

for every $r < R$, where

$$A(K, \alpha, \beta) = \frac{\Gamma(k+1)\,\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)\,\Gamma(k+\beta+1)}, \quad A(\alpha, \beta) = \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \qquad \ldots(2.4)$$

Combining (2.1) and (2.3) we get

$$\| H - g_n \|_{R_0} \leqslant \frac{M(r, H)}{\Gamma(q+1)} \sqrt{A(\alpha, \beta)} \sum_{k=n+1}^{\infty} \frac{\Gamma(k+q+1)}{\Gamma(k+1)}$$

$$\times \left[(2k + \alpha + \beta + 1)\, A(k, \alpha, \beta)\right]^{1/2} (R_0/r)^{2k}. \qquad \ldots(2.5)$$

Since $\Gamma(x + a)/\Gamma(x) \sim x^a$ as $x \to \infty$, we have

$$\frac{\Gamma(k+q+1)}{\Gamma(k+1)} \left((2k+\alpha+\beta+1)\, A(k, \alpha, \beta)\right)^{1/2} \sim \sqrt{2}\, k^{q+1/2}$$

as $k \to \infty$ and so

$$\frac{\Gamma(k+q+1)}{\Gamma(k+1)} \left((2k+\alpha+\beta+1)\, A(k, \alpha, \beta)\right)^{1/2} < 2\sqrt{2}\, k^{q+1/2}$$

for all $k > k_0$. Thus, for $n > k_0$ and $r > r^*$, where $r^* = (R + R_0)/2$ if $R < \infty$ and $r^* = 2R_0$ if $R = \infty$, using (2.5) and the above inequality, we get

$$\| H - g_n \|_{R_0} \leqslant \frac{M(r, H)}{(q+1)}\, 2\sqrt{2A(\alpha, \beta)} \sum_{k=n+1}^{\infty} k^{q+1/2} (R_0/r)^{2k}$$

$$\leqslant \frac{M(r, H)}{(q+1)}\, 2\sqrt{2A(\alpha, \beta)}\, (n+1)^{q+1/2} (R_0/r)^{2(n+1)}$$

$$\times \sum_{k=0}^{\infty} \left(1 + \frac{k}{k_0+1}\right)^{q+1/2} (R_0/r^*)^{2k}.$$

The lemma now follows from the above inequality.

Lemma 2—Let $H \in \bar{H}_{R_0}$. Then, for $n \geqslant 1$, we have

$$| a_n | R_0^{2n} \leqslant 2 ((2n+\alpha+\beta+1) A (\alpha, \beta) A (n, \alpha, \beta))^{1/2} E_{n-1} (H, R_0)$$

where $A (n, \alpha, \beta)$ and $A (\alpha, \beta)$ are given by (2.4).

PROOF : From the orthogonality of Jacobi polynomials[9] (p. 68) and the uniform convergence of the series (1.2) on $\bar{D}_{R_0}$ we have

$$a_n R_0^{2n} \Big/ ((2n+\alpha+\beta+1) A (n, \alpha, \beta))$$

$$= 2 \int_0^{\pi/2} H (R_0, \theta) P_n^{(\alpha,\beta)} (\cos 2\theta) \sin^{2\alpha+1} \theta \cos^{2\beta+1} \theta \, d\theta.$$

Thus, for any $g \in \pi_{n-1}$ we have

$$a_n R_0^{2n} \Big/ ((2n+\alpha+\beta+1) A (n, \alpha, \beta))$$

$$= 2 \int_0^{\pi/2} (H (R_0, \theta) - g (R_0, \theta)) P_n^{(\alpha,\beta)} (\cos 2\theta) \sin^{2\alpha+1} \theta \cos^{2\beta+1} \theta \, d\theta. \quad \ldots(2.6)$$

Using Schwartz's inequality and the orthogonality of Jacobi polynomials, the above relation gives

$$| a_n | R_0^{2n} \leqslant \|H - g\|_{R_0} ((2n+\alpha+\beta+1) A (n, \alpha, \beta) A (\alpha, \beta))^{1/2}. \quad \ldots(2.7)$$

By the definition of $E_n (H, R_0)$ there exists a GBSP polynomial $\tilde{g} \in \pi_{n-1}$ such that

$$\| H - \tilde{g} \|_{R_0} \leqslant 2E_{n-1} (H, R_0). \quad \ldots(2.8)$$

Taking, in particular $g = \tilde{g}$ in (2.7) and then using (2.8), we obtain the lemma from (2.7).

Using Lemmas 1 and 2 we now prove :

Theorem 1—Let $H \in \bar{H}_{R_0}$. Then $H \in H_R$, $R > R_0$, if and only if,

$$\limsup_{n\to\infty} (E_n (H, R_0))^{1/2n} = R_0/R.$$

PROOF : First, suppose that $H \in H_R$. Then using (1.5) and Lemma 1 we get

$$E_n(H, R_0) \leqslant \|H-g_n\|_{R_0} \leqslant K\, M(r, H)\,(n+1)^{q+1/2}\,(R_0/r)^{2(n+1)}. \qquad \ldots(2.9)$$

The above relation gives that

$$\limsup_{n\to\infty}\,(E_n(H, R_0))^{1/2n} \leqslant R_0/r$$

for all r sufficiently near to R and so

$$\limsup_{n\to\infty}\,(E_n(H, R_0))^{1/2n} \leqslant R_0/R.$$

On the other hand, using (2.2) and Lemma 2 we get

$$\left|\sum_{n=0}^{\infty} a_n r^{2n} P_n^{(\alpha,\beta)}(\cos 2\theta)\right| \leqslant |a_0| + \sum_{n=1}^{\infty} |a_n|\, r^{2n}\, \frac{\Gamma(n+q+1)}{\Gamma(n+1)\,\Gamma(q+1)}$$

$$\leqslant |a_0| + \frac{2\sqrt{A(\alpha,\beta)}}{\Gamma(q+1)} \sum_{n=0}^{\infty} E_{n-1}(H, R_0)\left(\frac{r}{R_0}\right)^{2n}$$

$$\times\,((2n+\alpha+\beta+1)\,A(n,\alpha,\beta))^{1/2}\,\frac{\Gamma(n+q+1)}{\Gamma(n+1)}. \qquad \ldots(2.10)$$

It follows from (2.10) that if $\limsup_{n\to\infty}(E_n(H, R_0))^{1/2n} < R_0/R$ then the series (1.2) converges uniformly on compact subsets of D_R, for some $R' > R$ and so, H is regular in $D_{R'}$, which is a contradiction. Hence, $\limsup_{n\to\infty}(E_n(H, R_0))^{1/2n} = R_0/R$. This proves the necessity part of the problem.

Sufficiency part can also be proved similarly. This proves the theorem.

For proving our next theorems we need the concepts of order and type of a function of a single complex variable u analytic in the disc $|u| < R$.

Let $f(u)$ be analytic in $|u| < R$, $0 < R < \infty$. Then the order ρ_0 of $f(u)$ is defined as

$$\rho_0 = \limsup_{r\to R} \frac{\log^+ \log^+ m(r, f)}{\log(R/(R-r))}$$

where

$$m(r, f) = \max_{|u|\leqslant r} |f(u)|$$

and if $0 < \rho_0 < \infty$ then the type T_0 of $f(u)$ is defined as

$$T_0 = \limsup_{r \to R} \frac{\log^+ m(r, f)}{(R/(R-r))^{\rho_0}} .$$

We now have :

Lemma 3 (MacLane[6], p. 47; Beuermann[2])—Let $f(u) = \sum_{n=0}^{\infty} b_n u^n$ be analytic in $|u| < R$ $(0 < R < \infty)$ and has order ρ_0 $(0 \leqslant \rho_0 \leqslant \infty)$. Then,

$$\rho_0 = \limsup_{n \to \infty} \frac{\log^+ \log^+ |b_n| R^n}{\log n - \log^+ \log^+ |b_n| R^n} .$$

Lemma 4 (Kapoor[5], p. 256; Bajpai *et al.*[1])—Let $f(u) = \sum_{n=0}^{\infty} b_n u^n$ be analytic in $|u| < R$ $(0 < R < \infty)$. Then f is of order ρ_0 $(0 < \rho_0 < \infty)$ and type T_0, if and only if,

$$\nu_0 = \frac{(\rho_0 + 1)^{\rho_0 + 1}}{\rho_0^{\rho_0}} T_0$$

where

$$\nu_0 = \limsup_{n \to \infty} \frac{(\log^+ |b_n| R^n)^{\rho_0 + 1}}{n^{\rho_0}}$$

satisfies $0 < \nu_0 < \infty$.

We now prove the main theorems of this section.

Theorem 2—Let $H \in H_R$, $0 < R < \infty$, be of order ρ and $R_0 < R$. Then

$$\rho = \limsup_{n \to \infty} \frac{\log^+ \log^+ (E_n(H, R_0)(R/R_0)^{2n})}{\log n - \log^+ \log^+ (E_n(H, R_0)(R/R_0)^{2n})} . \quad \ldots(2.11)$$

PROOF : Let the limit superior on the right hand side of (2.11) be denoted by d. Obviously $0 \leqslant d \leqslant \infty$. First let $0 < d < \infty$ and d' be an arbitrary number such that $0 < d' < d$. Then, by the definition of d there exists a sequence $\{n_k\}$ of positive integers tending to ∞ such that

$$\log (E_{n_k}(H, R_0)(R/R_0)^{2n_k}) > n_k^{d'/(1+d')} \quad \ldots(2.12)$$

for $k = 1, 2, 3, \ldots$. Using (2.9) and (2.12) we get

$$\log M(r, H) \geqslant n_k^{d'/(1+d')} + 2n_k \log (r/R) - (q+1)/2) \log (n_k+1) - \log K \qquad \ldots(2.13)$$

for all sufficiently large values of k and all r sufficiently near to R. Let $\{r_k\}$ be a sequence defined by

$$n_k = \{3 \log (R/r_k)\}^{-(1+d')} \qquad \ldots(2.14)$$

then $r_k \to R$ as $k \to \infty$. Now, using (2.13) and (2.14), for all sufficiently large values of k, we have

$$\log M(r_k, H) \geqslant \frac{1}{3^{1+d'}} (\log (R/r_k))^{-d'} + (q+1/2)(1+d') \log (3 \log (R/r_k)) - \log K + o(1).$$

Since $\log (R/(R - r_k)) \sim - \log \log (R/r_k)$ as $k \to \infty$, the above relation gives that

$$\limsup_{k \to \infty} \frac{\log^+ \log^+ M(r_k, H)}{\log (R/(R - r_k))} \geqslant d'.$$

Since d' $(< d)$ is arbitrary, this in turn gives that

$$\rho \geqslant d. \qquad \ldots(2.15)$$

Obviously, (2.15) holds for $d = 0$. For $d = \infty$ the above arguments give that $\rho = \infty$.

On the other hand, using (2.10) we get

$$M(r, H) \leqslant |a_0| + \frac{2\sqrt{A(\alpha, \beta)}}{\Gamma(q+1)} m(r, h) \qquad \ldots(2.16)$$

where, by Theorem 1,

$$h(u) = \sum_{n=1}^{\infty} ((2n+\alpha+\beta+1) A(n, \alpha, \beta))^{1/2} \frac{\Gamma(n+q+1)}{\Gamma(n+1)} \times E_{n-1}(H, R_0)(u/R_0)^{2n} \qquad \ldots(2.17)$$

is analytic in $|u| < R$. Using (2.16) and applying Lemma 3 to $h(u)$ we get

$$\rho \leqslant d. \qquad \ldots(2.18)$$

Theorem 2—now follows from (2.15) and (2.18).

Theorem 3—Let $H \in H_R$, $0 < R < \infty$, and $R_0 < R$. Then H is of order ρ $(0 < \rho < \infty)$ and type T, if and only if

$$\nu = (\rho+1)^{\rho+1} (2/\rho)^{\rho} T$$

where

$$\nu = \limsup_{n\to\infty} \frac{(\log^+ (E_n (H, R_0) (R/R_0)^{2n}))^{\rho+1}}{n^\rho} \qquad ...(2.19)$$

satisfies $0 < \nu < \infty$.

PROOF : (i) First, let $H \in H_R$ be of order ρ and type T $T (< \infty)$. Then, for $\epsilon > 0$, (1.8) gives that there exists $r_0 = r_0 (\epsilon)$ such that

$$\log M (r, H) \leqslant (T + \epsilon) (R/(R - r))^\rho \qquad ...(2.20)$$

for $r_0 < r < R$. Using (2.9) and (2.20) we get

$$\log^+ (E_n (H, R_0) (R/R_0)^{2n}) \leqslant (T + \epsilon) (R/R - r))^\rho + 2n \log (R/r)) + (q+1/2) \log (n+1) + \log^+ K \qquad ...(2.21)$$

for all r sufficiently near to R and all sufficiently large values of n. Choose a sequence $\{r'_n\}$ as

$$R/(R - r'_n) = \left(\frac{2n}{\rho (T+\epsilon)} \right)^{1/(1+\rho)}. \qquad ...(2.22)$$

Clearly $r'_n \to R$ as $n \to \infty$. Using (2.21) and (2.22) we get

$$\log^+ (E_n (H, R_0) (R/R_0)^{2n}) \leqslant \frac{(T+\epsilon)^{1/(1+\rho)} (2n)^{\rho/(1+\rho)}}{\rho^{\rho/(1+\rho)}} (1+\rho+o(1))$$

for all sufficiently large values of n. On proceeding to limits, the above inequality gives that

$$\nu \leqslant (\rho + 1)^{\rho+1} (2/\rho)^\rho T. \qquad ...(2.23)$$

The reverse inequality in (2.23) follows by using (2.16) and applying Lemma 4 to the function $h(u)$ given by (2.17). This proves the necessity part of the theorem.

(ii) Suppose that $0 < \nu < \infty$. Then (2.19) gives that

$$\rho = \limsup_{n\to\infty} \frac{\log^+ \log^+ (E_n (H, R_0) (R/R_0)^{2n})}{\log n - \log^+ \log^+ (E_n (H, R_0) (R/R_0)^{2n})}$$

and so, by Theorem 2, H is of order ρ. Sufficiency part now follows from the necessity part of the theorem.

This proves the theorem.

3. THE ERRORS $E^1_{n,\delta} (H, R_0)$, $E^2_{n,\delta} (H, R_0)$ AND THE COEFFICIENTS a_n

In this section we obtain some results analogous to Theorems 1, 2 and 3 for the approximation errors in L^δ-norms and the coefficients a_n in (1.2). Since the techniques

used in proving these results are the same as those used in Section 2 we o nly give the outlines of the proofs.

Let $H \in H_R$ and $R_0 < R$. Using Lemma 1, (1.3), (1.4), (1.6) and (1.7) we get

$$E^{i}_{n,\delta}(H, R_0) \leqslant K_i\,(n+1)^{q+1/2}\,(R_0/r)^{2(n+1)}\,M(r, H);\ i = 1, 2 \qquad \ldots(3.1)$$

for all r sufficiently near to R and all sufficiently large values of n. Here K_1 is a constant depending on R_0, w and δ only and K_2 a constant depending on R_0, $\bar{w}$ and δ.

On the other hand, by (1.6), for $H \in \bar{H}_{R_0}$ there exists a GBSP polynomial $g^*_{n-1} \in \pi_{n-1}$ such that

$$2E^1_{n-1,\delta}(H, R_0) \geqslant \|H - g^*_{n-1}\|^{1}_{R_0,\delta}$$

$$\geqslant \frac{1}{T^{1/\delta}}\left(\int_0^{2\pi} |\,H(R_0, \theta) - g^*_{n-1}(R_0, \theta)\,|^{\delta}\,d\theta\right)^{1/2} \quad \ldots(3.2)$$

since $1/w$ is bounded we have $w \geqslant 1/T$, $T > 0$. For $\delta > 1$ choose η such that $(1/\delta) + (1/\eta) = 1$. Using Holder's inequality we get

$$\int_0^{2\pi} |\,H(R_0, \theta) - g^*_{n-1}(R_0, \theta)\,|\,d\theta$$

$$\leqslant \left(\int_0^{2\pi} |\,H(R_0, \theta) - g^*_{n-1}(R_0,\theta)\,|^{\delta}\,d\theta\right)^{1/\delta}\left(\int_0^{2\pi} d\theta\right)^{1/\eta}. \qquad \ldots(3.3)$$

Combining (3.2) and (3.3) we get

$$2E^1_{n-1,\delta}(H, R_0) \geqslant \frac{1}{T^{1/\delta}(2\pi)^{1/\eta}}\int_0^{2\pi} |\,H(R_0, \theta) - g^*_{n-1}(R_0, \theta)\,|\,d\theta$$

$$= \frac{4}{T^{1/\delta}(2\pi)^{1/\eta}}\int_0^{\pi/2} |\,H(R_0, \theta) - g^*_{n-1}(R_0, \theta)\,|\,d\theta \quad \ldots(3.4)$$

for $\delta > 1$, since GBSP's H and g^*_{n-1} are even in x and y. For $\delta = 1$, (3.4) is obvious with $\eta = \infty$. Taking, in particular, $g = g^*_{n-1}$ in (2.6) and using (2 2) we get

$$| a_n | R_0^{2n} / ((2n + \alpha + \beta + 1) A(n, \alpha, \beta))$$

$$\leqslant \frac{2\Gamma(n+q+1)}{\Gamma(q+1)\Gamma(n+1)} \int_0^{\pi/2} | H(R_0, \theta) - g_{n-1}^{*}(R_0, \theta) | \, d\theta. \qquad \ldots(3.5)$$

Combining (3.4) and (3.5) we obtain

$$| a_n | R_0^{2n} \leqslant \frac{T^{1/\delta} (2\pi)^{1/\eta} (2n+\alpha+\beta+1) A(n, \alpha, \beta) \Gamma(n+q+1)}{\Gamma(q+1)\Gamma(n+1)}$$

$$\times E^1_{n-1,\delta}(H, R_0). \qquad \ldots(3.6)$$

Using (3.1) with $i = 1$ and (3.6) we see that $E_n(H, R_0)$ can be replaced by $E^1_{n,\delta}(H, R_0)$ in Theorems 1, 2 and 3.

Similarly, by (1.7), for $H \in \bar{H}_{R_0}$, there exists $\bar{g}_{n-1} \in \pi_{n-1}$ such that

$$2E^2_{n-1,\delta}(H, R_0) \geqslant \|H - \bar{g}_{n-1}\|^2_{R_0,\delta}$$

$$\geqslant \frac{1}{\bar{T}^{1/\delta}} \left(\iint_{\bar{D}_{R_0}} \left| H(x, y) - \bar{g}_{n-1}(x, y) \right|^{\delta} dx\, dy \right)^{1/\delta}$$

$$\geqslant \frac{1}{\bar{T}^{1/\delta} (\pi R_0^2)^{1/\eta}}$$

$$\times \iint_{\bar{D}_{R_0}} | H(x, y) - \bar{g}_{n-1}(x, y) | \, dx\, dy \qquad \ldots(3.7)$$

where $w \geqslant 1/\bar{T}$, $\bar{T} > 0$ and $(1/\delta) + (1/\eta) = 1$. From the orthogonality of Jacobi polynomials and uniform convergence of the series (1.2) on $\bar{D}_{R_0}$ we have

$$a_n r^{2n} / ((2n+\alpha+\beta+1) A(n, \alpha, \beta))$$

$$= 2 \int_0^{\pi/2} (H(r, \theta) - \bar{g}_{n-1}(r, \theta)) \, P_n^{(\alpha,\beta)}(\cos 2\theta) \sin^{2\alpha+1}\theta \cos^{2\beta+1}\theta \, d\theta$$

for $0 < r \leqslant R_0$. Using (2.2) we get

$$| a_n | r^{2n} / ((2n+\alpha+\beta+1) A(n, \alpha, \beta))$$

$$\leqslant \frac{\Gamma(n+q+1)}{2\Gamma(q+1)\Gamma(n+1)} \int_0^{2\pi} | H(r, \theta) - \bar{g}_{n-1}(r, \theta) | \, d\theta$$

since H and $\bar{g}_{n-1}$ are even in x and y. Multiplying both sides of the above inequality by $r\,dr$ and integrating from 0 to R_0 we get

$$| a_n | \; R_0^{2n+2} /((2n+2)(2n+\alpha+\beta+1)\, A(n, \alpha, \beta))$$

$$\leqslant \frac{\Gamma(n+q+1)}{2\Gamma(q+1)\,\Gamma(n+1)} \iint\limits_{\bar{D}_{R_0}} | H(x,y) - \bar{g}_{n-1}(x,y) | \, dx\, dy. \qquad ...(3.8)$$

Combining (3.7) and (3.8) we obtain

$$| a_n | \; R_0^{2n+2} \leqslant \frac{\bar{T}^{1/\delta} (\pi R_0^2)^{1/\eta} (2n+2)(2n+\alpha+\beta+1)\, A(n,\alpha,\beta)\, \Gamma(n+q+1)}{\Gamma(q+1)\,\Gamma(n+1)}$$

$$\times \; E^2_{n-1,\delta}(H, R_0). \qquad ...(3.9)$$

Using (3.1) with $i = 2$ and (3.9) we see that $E_n(H, R_0)$ can be replaced by $E^2_{n,\delta}(H, R_0)$ in Theorems 1, 2 and 3.

Finally, let $H \in H_R$. Then, for $r < R$, using (2.10) we get

$$M(r, H) \leqslant | a_0 | + m(r, \widetilde{h})/\Gamma(q+1) \qquad ...(3.10)$$

where, by Theorem 1 of Fryant[3]

$$\widetilde{h}(u) = \sum_{n=1}^{\infty} \frac{\Gamma(n+q+1)}{\Gamma(n+1)} \, | a_n | \, u^{2n}$$

is analytic in $| u | < R$.

Using (2.3) and (3.10) we obtain the following coefficient characterizations of order and type of $H \in H_R$, $0 < R < \infty$:

Theorem 4—Let $H \in H_R$, $0 < R < \infty$, be of order ρ. Then

$$\rho = \limsup_{n\to\infty} \frac{\log^+ \log^+ | a_n | \, R^{2n}}{\log n - \log^+ \log^+ | a_n | \, R^{2n}} .$$

Theorem 5—Let $H \in H_R$, $0 < R < \infty$. Then, H is of order ρ $(0 < \rho < \infty)$ and type T, if and only if

$$\nu_* = (\rho+1)^{\rho+1} (2/\rho)^\rho \, T$$

where

$$\nu_* = \limsup_{n\to\infty} \frac{(\log^+ | a_n | \, R^{2n})^{\rho+1}}{n^\rho}$$

satisfies $0 < \nu_* < \infty$.

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ABSOLUTE SUMMABILITY FACTORS

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In this paper, the author has investigated sufficient condition, which cannot be relaxed to any appreciable extent, for absolute summability factors for $|R, \exp\{n/\log(n+1)\}, 1|$, for both general infinite series and Fourier series, which consequently improves the corresponding results due to Lal [*Proc. Am. Math. Soc.* 14 (1963), [311-19]. The class of functions, considered by the present author, for the Fourier series strictly contains the class of functions considered by Lal.[8]

1. DEFINITIONS AND NOTATIONS

Let f be 2π-periodic and L-integrable over $(-\pi, \pi)$ and let, without loss of generality, the Fourier series of f at x be

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} A_n(x).$$

Let x be any fixed real number. Then we write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\} \qquad \ldots(1.1)$$

$$\phi_1(t) = t^{-1}\int_0^t \phi(u)\,du \qquad \ldots(1.2)$$

$$P(t) = \phi(t) - \phi_1(t) \qquad \ldots(1.3$$

$$P_1(t) = \phi_1(t) - t^{-1}\int_0^t \phi_1(u)\,du \qquad \ldots(1.4)$$

$$e(n) = \exp\{n/\log(n+1)\}\ (n \geq 1) \qquad \ldots(1.5)$$

$$\epsilon_n = n^{-1} y_n \log(n+1) \qquad \ldots(1.6)$$

$$\Delta z_n = z_n - z_{n+1} \text{ (for any sequence } (z_n)). \qquad \ldots(1.7)$$

We also write $T = [2\pi/t]$, the greatest integer less than or equal to $(2\pi/t)$.

It is known that the summability methods $|R, e(n), 1|$ and $|R', e(n), 1|$ are equivalent (see Mohanty[9], footnote to the page 298).

A series $\sum_{n=1}^{\infty} d_n \in | R, e(n), 1 |$ if and only if (see Chandra[2])

$$\sum_{n=1}^{\infty} \Delta (1/e(n)) \left| \sum_{m=1}^{n} e(m) d_m \right| < \infty . \qquad \ldots(1.8)$$

Let $L(-\pi, \pi)$ be the set of all 2π-periodic and L-integrable function f. Then we write

$$S(f; \phi_1) = \{ f \in L(-\pi, \pi) : \phi_1(t) \in BV(0, \pi) \}.$$

Given a function h, we also write

$$C(f; P_1 : h)$$

for the set of all functions $f \in L(-\pi, \pi)$ for which

$$\lim_{y \to 0+} \int_{y}^{t} \phi_1(u)\, du \qquad \ldots(1.9)$$

exists (in the sense of Cauchy) and

$$(P_1(t)/h(t^{-1})) \in BV(0, \pi). \qquad \ldots(1.10)$$

In the case $h(t^{-1}) = 1$ for all t, we simply write $C(f; P_1)$ for $C(f; P_1; 1)$.

2. Introduction

Lal[8] proved the following :

Theorem A—Let the sequence (y_n) be

convex, and $\qquad \ldots(2.1)$

$$\sum_{n=1}^{\infty} n^{-1} | y_n | < \infty . \qquad \ldots(2.2)$$

Then

$$f \in S(f; \phi_1) \Rightarrow \sum_{n=1}^{\infty} A_n(x) \epsilon_n \in | N, 1/(n+1) | . \qquad \ldots(2.3)$$

In an attempt to prove Theorem A, Lal[8] also proved the following :

Theorem B—Let (y_n) satisfy (2.1) and (2.2) and let

$$\sum_{m=1}^{n} m\, d_m = O(n). \qquad \ldots(2.4)$$

Then

$$\sum_{n=1}^{\infty} d_n \epsilon_n \in | N, 1/(n+1) | .$$

We first observe that if (y_n) satisfies (2.1) and (2.2) then

$$0 \leqslant \Delta y_n \downarrow, \qquad \ldots(2.5)$$

$$\sum_{n=1}^{\infty} \log (n+1) \Delta y_n < \infty . \qquad \ldots(2.6)$$

For (2.5), see Zygmund[11]; p. 93 and Chow[5] and for (2.6), see Mohapatra[10]. Also from Lemma 1 of the present paper it follows that the summability method $| R, e(n), 1 |$ is weaker than $| N, 1/(n+1) |$. Thus, in view of (2.5) and (2.6) which are consequences of (2.1) and (2.2), we replace (2.1) by

$$\sum_{n=1}^{\infty} n^{-1} | \Delta y_n | \log | (n+1) < \infty \qquad \ldots(2.7)$$

and the method $| N, 1/(n+1) |$ by $| R, e(n), 1 |$ to improve Theorem B in the following form :

Theorem 1—Let (y_n) be a given sequence satisfying (2.7). In order that (y_n) should be such that, for every (d_n) satisfying (2.4),

$$\sum_{n=1}^{\infty} d_n \epsilon_n \in | R, e(n), 1 | . \qquad \ldots(2.8)$$

it is sufficient that (2.2) should hold.

Remark 1: The following example shows that (2.2) cannot be relaxed to any appreciable extent :

Example—Let

$$y_n = 1/\log (n+1) \text{ and } d_n = (-1)^n \; (n \geqslant 1).$$

Then (2.2) does not hold but (2.7) holds. Also

$$\sum_{m=1}^{n} m \, dm = \sum_{m=1}^{n} (-1)^m m = O(n),$$

which proves that (2.4) holds. Now to prove that (2.8) does not hold, it is sufficient to show that, in view of (3.1),

$$\sum_{n=1}^{\infty} d_n \epsilon_n \notin | N, 1/(n+1) | .$$

Suppose, on the contrary, that

$$\sum_{n=1}^{\infty} d_n \epsilon_n = \sum_{n=1}^{\infty} (-1)^n n^{-1} \in | N, 1/(n+1) |$$

then we must have (see Corollary to Theorem 1 of Das[6])

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n \log (n+1)} \right| < \infty ,$$

which is not true. Therefore (2.8) does not hold.

This proves that if you relax the condition (2.2) to some appreciable extent then there exist sequences (d_n) and (y_n) satisfying (2.4) and (2.7) respectively for which (2.8) does not hold.

Now, in view of Lemmas 1 and 2, we improve Theorem A, by proving

Theorem 2—Let (y_n) be a given sequence satisfying (2.7). Then

$$f \in C(f, P_1) \Rightarrow \sum_{n=1}^{\infty} A_n(x) \epsilon_n \in | R, e(n), 1 | , \quad \text{..(2.9)}$$

provided (2.2) holds.

This may be deduced from the following general theorem. However, a simpler proof of Theorem 2 may be obtained by combining the arguments of Remark 2 (given after Theorem 3) and Theorem 1.

Theorem 3—Let (y_n) be a given sequence satisfying (2.7) and let there exist a function $h(1/t) \geqslant 0$ such that, uniformly in $0 < t \leqslant \pi$,

$$\frac{d}{dt}(t\, h(1/t)) = O\{h(1/t)\} \quad \text{...(2.10)}$$

$$t\, h(1/t) \log (\pi/t) = O(1) \quad \text{...(2.11)}$$

$$h(1/t) \sum_{n=T}^{\infty} n^{-1} | y_n | = O(1) \quad \text{...(2.12)}$$

$$h(1/t) \sum_{n=T}^{\infty} n^{-1} | \Delta y_n | \log (n+1) = O(1) \quad \text{...(2.13)}$$

$$h(1/t) \sum_{n=1}^{T} n^{-1} | \Delta y_n | e(n) \log (n+1) = O\{e(T)\}. \quad \text{...(2.14)}$$

We further suppose that as t increases

$$0 \leqslant t^{\beta} \frac{d}{dt}\left\{ t \ \frac{d}{dt}\ (t\ h\ (1/t))\right\} \text{ is non-decreasing for some } \beta \text{ with}$$

$$0 \leqslant \beta < 1 \qquad \ldots(2.15)$$

$$t^{-1}\ \frac{d}{dt}\left\{ t\ \frac{d}{dt}\ (t\ h\ (1/t))\right\} \text{ is non-increasing} \qquad \ldots(2.16)$$

and

$$\left[\frac{d}{dt}\left\{ t\ \frac{d}{dt}\ (t\ h\ (1/t))\ \right\}\right]_{t=1/n} = O\ \{h\ (n)\}. \qquad \ldots(2.17)$$

Then, since (2.2) is included in (2.12) we have

$$f \in C\ (f;\ P_1;\ h)\ \Rightarrow \sum_{n=1}^{\infty} A_n\ (x)\ \epsilon_n \in |\ R,\ e\ (n),\ 1\ |\ . \qquad \ldots(2.18)$$

Remark 2 : Suppose h is such that $C\ (f;\ P_1) \subset C\ (f;\ P_1;\ h)$ then condition (2.2) cannot be relaxed to any appreciable extent. To prove it, we have as in Theorem 2 of Chandra and Gupta[8].

$$A_n\ (x) = 2P_1\ (\pi) \cos n\pi + \frac{2}{\pi}\int_0^{\pi} P_1\ (t)\ t\ \frac{d}{dt}\left(\frac{\sin nt}{nt} - \cos nt\right) dt.$$

Now, we take $f \in C\ (f;\ P_1)$, then $f \in C\ (f;\ P_1;\ h)$. However, for $f \in C(f;\ P_1)$,

$$A_n\ (x) = O\ (n^{-1}) + \frac{2}{\pi}\int_0^{\pi} t \cos nt\ dP_1\ (t)$$

and hence

$$\sum_{m=1}^{n} m\ A_m\ (x) = O\ (n) + \frac{2}{\pi}\int_0^{\pi} t\ dP_1\ (t)\left(\sum_{m=1}^{n} m \cos mt\right)$$

$$= O\ (n),$$

which proves that (2.4) holds with $d_m = A_m\ (x)$. Thus by taking $d_n = A_n\ (x)$ in the example of Remark 1, if follows that (2.18) does not hold.

3. Lemmas

We require the following lemmas for the proof of the theorems :

Lemma 1—The inclusion

$$|\ N,\ 1/(n+1)\ | \supset |\ R,\ \exp\ \{n/\log\ (n+1)\},\ 1\ | \qquad \ldots(3.1)$$

holds. However, for $0 < \beta \leqslant 1$, the inclusion

$$| N, 1/(n+1) | \subset | R, \exp \{n/(\log(n+1))^{\beta}\}, 1 | \qquad \ldots(3.2)$$

is false.

For the proof of (3.1), see Dikshit[7] and for (3.2), see Bosanquet and Das[1]: Theorem J.

Lemma 2—The inclusion

$$S(f; \phi_1) \subset C(f; P_1) \qquad \ldots(3.3)$$

is strict.

PROOF: The inclusion (3.3) is simple to get. Thus we prove that the inclusion is strict. To prove it we consider an even function $f \in L(-\pi, \pi)$ so that at $x = 0$,

$$\phi(t) = f(t).$$

Now we define

$$f(t) = \begin{cases} \dfrac{d}{dt}(t \log(2\pi/t)) & (0 < t \leqslant \pi) \\ 0 & (t = 0)\,. \end{cases} \qquad \ldots(3.4)$$

Then

$$\phi_1(t) = \log(2\pi/t)$$

and hence $f \notin S(f; \phi_1)$. However, for f defined by (3.4), we obtain that

$$P_1(t) = -1.$$

This proves that $f \in C(f; P_1)$ and hence the lemma follows.

Lemma 3—Let that function $h(1/t)$ satisfy (2.15) through (2.17). Then, uniformly in $0 < t \leqslant \pi$,

$$\int_0^t \frac{\sin nu}{nu} \frac{d}{du}\left\{ u \frac{d}{du}(u\, h(1/u))\right\} du = O\{n^{-1} h(n)\}.$$

This may be obtained by using the technique of Lemma 3 of Chandra and Mohapatra[4].

4. Proof of Theorem 1

The series $\sum_{n=1}^{\infty} d_n \epsilon_n \in | R, e(n), 1 |$ if

$$\Sigma = \sum_{n=1}^{\infty} \Delta(1/e(n)) \left| \sum_{m=1}^{\infty} e(m)\, d_m\, m^{-1} y_m \log(m+1) \right| < \infty.$$

Now, by Abel's transformation

$$\sum_{m=1}^{n} e(m)\, d_m\, m^{-1} y_m \log(m+1) = \sum_{m=1}^{n-1} \Delta y_m \sum_{k=1}^{m} k^{-1} e(k)\, d_k \log(k+1)$$

$$+ y_n \sum_{k=1}^{n} k^{-1} e(k)\, d_k \log(k+1) \quad \ldots(4.1)$$

And the sequence $(n^{-2} e(n) \log(n+1))$ is ultimately increasing with n, therefore by Abel's lemma

$$\left| \sum_{m=1}^{n} m^{-1} e(m)\, d_m \log(m+1) \right| = O\{n^{-1} e(n) \log(n+1)\} \qquad \ldots(4.2)$$

whenever (2.4) holds. Hence by using (4.1) and (4.2) in Σ we get

$$\Sigma = O(1) \sum_{n=1}^{\infty} \Delta(1/e(n)) \sum_{m=1}^{n} |\Delta y_m|\, m^{-1} e(m) \log(m+1)$$

$$+ O(1) \sum_{n=1}^{\infty} \Delta(1/e(n))\, n^{-1} e(n)\, y_n(v) \log(n+1)$$

$$= \Sigma_1 + \Sigma_2, \text{ say.}$$

By changing the order of summation, we get

$$\Sigma_1 = O(1) \sum_{m=1}^{\infty} |\Delta y_m|\, m^{-1} \log(m+1) < \infty,$$

by (2.7), and using the inequality

$$|\Delta e(n)| \log(n+1) = O\{e(n+1)\}$$

we obtain that

$$\Sigma_2 = O(1) \sum_{n=1}^{\infty} n^{-1} |y_n| < \infty,$$

by (2.2).

This completes the proof of Theorem 1.

5. Proof of Theorem 3

Proceeding as in Theorem 2 of Chandra and Gupta[3], we get

$$A_n(x) = \frac{2}{\pi} \int^{\pi} \{P_1(t)/h(t^{-1})\}\, h(t^{-1})\, t \frac{d}{dt}\left(\frac{\sin nt}{nt} - \cos nt\right) dt$$

$$+ 2 P_1(\pi) \cos n\pi$$

$$= -\frac{2}{\pi}\frac{P_1(\pi)}{h(1/\pi)}\int_0^\pi \frac{\sin nt}{nt}\frac{d}{dt}\left(t\frac{d}{dt}(t\,h(1/t))\right)dt$$

$$-\frac{2}{\pi}\int_0^\pi d\,\{P_1(t)/h(1/t)\}\left[t\,h(1/t)\left(\frac{\sin nt}{nt}-\cos nt\right)\right.$$

$$\left.+\frac{\sin nt}{n}\frac{d}{dt}(t\,h(1/t))-\int_0^t \frac{\sin nu}{nu}\frac{d}{du}\left\{u\frac{d}{du}(u\,h(1/u))\right\}du\right], \qquad \ldots(5.1)$$

since, by (2.10), (2.11) and (2.15), we obtain that for $0 < t \leqslant \pi$,

$$\int_0^t u\,h(1/u)\frac{d}{du}\left(\frac{\sin nu}{nu}-\cos nu\right)du$$

$$= t\,h(1/t)\left(\frac{\sin nt}{nt}-\cos nt\right)+\int_0^t \frac{d}{du}\left(\frac{\sin nu}{nu}\right)u\frac{d}{du}(u\,h(1/u))\,du$$

$$= t\,h(1/t)\left(\frac{\sin nt}{nt}-\cos nt\right)+\frac{\sin nt}{n}\frac{d}{dt}(t\,h(1/t))$$

$$-\int_0^t \frac{\sin nu}{nu}\frac{d}{du}\left\{u\frac{d}{du}(u\,h(1/u))\right\}du.$$

Now, by (2.18), $\int_0^\pi d\,|\,\{P_1(t)/h(t^{-1})\}\,| < \infty$, therefore by Lemma 3, we get

$$A_n(x) = O\{n^{-1}h(n)\}+\frac{2}{\pi}\int_0^\pi d\,\{P_1(t)/h(1/t)\}\,t\,h(1/t)\cos nt$$

$$-\frac{2}{\pi}\int_0^\pi d\,\{P_1(t)/h(1/t)\}\left\{h(1/t)+\frac{d}{dt}(t\,h(1/t))\right\}\frac{\sin nt}{n}$$

$$= A_n^{(1)}(x)+A_n^{(2)}(x)+A_n^{(3)}(x), \text{ say.}$$

However, by (2.11).

$$\sum_{n=1}^{\infty}\left|A_n^{(1)}(x)\,\epsilon_n\right| = O(1)\sum_{n=1}^{\infty}n^{-1}\,|\,y_n\,|,$$

which is finite by (2.2). Thus $\sum^{\infty} A_n^{(1)}(x)\,\epsilon_n$ is absolutely convergent. Hence by the absolute regularity of $|\, R, e\,(n), 1\,|$

$$\sum_{n=1}^{\infty} A_n^{(1)}(x)\,\epsilon_n \in |\, R, e\,(n), 1\,| .$$

Thus, whenever (2.10) holds and $f \in C\,(f;\, P_1;\, h)$, for the proof of the sufficiency part, it would be sufficient to show that

$$\sum_{n=1}^{\infty} \Delta\,(1/e\,(n))\,|\sum_{m=1}^{n} e\,(m)\,m^{-1}\,y_m \log\,(m+1)\,K\,(m, t)\,|$$

$$= (\{t\,h(1/t)\}^{-1}), \qquad \ldots(5.2)$$

uniformly in $0 < t \leqslant \pi.$, where $K\,(m, t)$ is either $\cos mt$ or $\dfrac{\sin\,mt}{mt}$.

We split up the sum $\sum\limits_{n=1}^{\infty}$ into $\sum\limits_{n\leqslant T}$ and $\sum\limits_{n>T}$. Then by using $|\,K\,(m, t)\,| \leqslant 1$ and changing the order of summation, we get

$$\sum_{n\leqslant T} \leqslant \sum_{m=1}^{T} m^{-1}\,y_m \log\,(m+1) = O\,\{[t\,h(1/t)]^{-1}\}, \qquad \ldots(5.3)$$

by (2.2) and (2.11), uniformly in $0 < t \leqslant \pi$.

Now, by Abel's transformation and Abel's lemma,

$$|\sum_{m=1}^{n} e\,(m)\,m^{-1}\,y_m \log\,(m+1)\,K\,(m, t)\,|$$

$$\leqslant \sum_{m=1}^{n} |\,\Delta\,y_m\,|\,|\sum_{r=1}^{m} e(r)\,r^{-1} \log\,(r+1)\,K\,(r, t)\,|$$

$$+\,|\,y_n\,|\,|\sum_{r=1}^{n} e(r)\,r^{-1} \log\,(r+1)\,K\,(r, t)\,|$$

$$= O\,(t^{-1}) \sum_{m=1}^{n} |\,\Delta\,y_m\,|\,e\,(m)\,m^{-1} \log\,(m+1)$$

$$+\,O\,(t^{-1})\,|\,y_n\,|\,e\,(n)\,n^{-1} \log\,(n+1),$$

since $(n^{-1}\,e(n) \log\,(n+1))$ ultimately increases with n and

$$\sum_{n=a}^{b} K\,(n, t) = O\,(1/t)\ (0 < a < b \leqslant \infty).$$

Hence

$$\sum_{n>T} = O(1/t) \sum_{n=T}^{\infty} \Delta(1/e(n)) \sum_{m=1}^{n} |\Delta y_m| \, e(m) \, m^{-1} \log(m+1)$$

$$+ O(1/t) \sum_{n=T}^{\infty} \Delta(1/e(n)) \, |y_n| \, e(n) \, n^{-1} \log(n+1) \qquad \ldots(5.4)$$

However

$$e(n) \log(n+1) \, \Delta(1/e(n)) = O(1),$$

therefore the second term on the right of (5.4) does not exceed

$$O(1/t) \sum_{n=T}^{\infty} n^{-1} \, |y_n| = O\{(t \, h(1/t))^{-1}\}, \qquad \ldots(5.5)$$

by (2.12) and the first term does not exceed

$$O(1/t) \sum_{n=T}^{\infty} \Delta(1/e(n)) \sum_{m=1}^{T} |\Delta y_m| \, m^{-1} e(m) \log(m+1)$$

$$+ O(1/t) \sum_{n=T}^{\infty} \Delta(1/e(n)) \sum_{m=T}^{n} |\Delta y_m| \, m^{-1} e(m) \log(m+1)$$

$$= O\{(t \, e(T))^{-1} \sum_{m=1}^{T} |\Delta y_m| \, m^{-1} e(m) \log(m+1)\}$$

$$+ O(1/t) \sum_{m=T}^{\infty} |\Delta y_m| \, m^{-1} \log(m+1)$$

$$= O\{(t \, h(1/t))^{-1}\}, \qquad \ldots(5.6)$$

uniformly in $0 < t \leqslant \pi$, by (2.14) and (2.13).

Thus combining (5.3) through (5.6), we get (5.2).

This completes the proof of Theorem 3.

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UNSTEADY MIXED CONVECTION LAMINAR BOUNDARY—LAYER FLOW OVER A VERTICAL PLATE IN MICROPOLAR FLUIDS

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The combined effect of forced and free convection on the unsteady laminar incompressible boundary-layer flow of a thermo-micropolar fluid over a semi-infinite vertical plate has been studied when the free-strem velocity, surface mass transfer and wall temperature vary arbitrarily with time. The partial differential equations with three independent variables governing the flow have been solved using quasilinearization in combination with an implicit finite-difference scheme. The results indicate that the buoyancy parameter, coupling parameter, mass transfer and unsteadiness in the free-stream velocity strongly affect the skin friction, microrotation gradient and heat transfer whereas the effect of microrotation parameter on the skin friction and heat transfer is rather weak, but microrotation gradient is strongly affected by it. The heat transfer is strongly dependent on the Prandtl number, the dissipation parameter and the variation of the wall temperature with time whereas the skin friction and microrotation gradient are weakly dependent on it. The buoyancy parameter causes an overshoot in the velocity profile. The magnitude of the velocity overshoot increases as the buoyancy parameter increases and it decreases as time increases.

1. INTRODUCTION

When the velocity of the fluid is small and the temperature difference between the surface and ambient fluid is large than the buoyancy effects on forced convective heat-transfer become important. The combined effect of forced and free convection over a heated vertical plate for Newtonian fluid has been studied by several investigators[1-4]. The flow and heat transfer behaviour of Polymeric fluids, colloidal fluids, real fluid with suspensions, liquid crystals and animal blood cannot be explained on the basis of Newtonian and non-Newtonian fluid theory. The theory of micropolar and thermomicropolar fluids was introduced by Eringen[5-7]. An excellent review of the micropolar theory is given by Ariman *et al.*[8-9]. Several investigators[10,14] have studied the steady forced or free convection boundary-layer flow for micropolar fluids. Recently, Jena and Mathur[15] have studied the steady mixed convection laminar boundary-layer flow of a micropolar fluid from a vertical plate without dissipation effects. It may be noted that the unsteady mixed convection flow over a vertical plate in a micropolar fluid has not been studied so far.

We have investigated the unsteady laminar incompressible boundary-layer flow of a micropolar fluid over a vertical flat plate when the free-stream velocity, mass transfer and the wall temperature vary arbitrarily with time. The effects of the surface mass transfer which varies arbitrarily with time, viscous dissipation and the Prandtl number have also been taken into account. The partial differential equations with three independent variables governing the flow have been solved numerically using a quasilinear finite-difference scheme. The results have been compared with Oosthuizen and Hart[3], Gryzagoridis[4] and Jena and Mathur[15].

2. Governing Equations

We consider the unsteady laminar incompressible boundary-layer flow of a thermomicropolar fluid past a vertical plate under the combined effect of forced and free convection. It has been assumed that the free-stream fluid temperature remains constant and the free-stream velocity, surface mass transfer and the wall temperature vary with time. Under the foregoing assumptions, the equations governing the flow can be written as[5,7,10-15]

$$u_x + v_y = 0 \qquad \text{...(1a)}$$

$$u_t + uu_x + vu_y = (u_e)_t + [(\mu + k_1)/\rho]\, u_{yy} + (k_1/\rho)\, N_y + g\beta\,(T - T_\infty) \qquad \text{...(1b)}$$

$$N_t + uN_x + vN_y = (\gamma/\rho j)\, N_{yy} - (k_1/\rho j)\,[2N + u_y] \qquad \text{...(1c)}$$

$$T_t + uT_x + vT_y = Pr^{-1}\,(\mu/\rho)\, T_{yy} + (\alpha_1/\rho c_p)\,[T_x N_y - T_y N_x] + (1/\rho c_p)\Big[(\mu + k_1/2)\, u_y^2 + 2k_1\,(N + u_y/2)^2 + \gamma N_y^2\Big]. \qquad \text{...(1d)}$$

The relevant initial and boundary conditions are

$$\left.\begin{aligned} &u\,(x, y, 0) = u_i\,(x, y),\ v\,(x, y, 0) = v_i\,(x, y) \\ &N\,(x, y, 0) = N_i\,(x, y),\ T\,(x, y, 0) = T_i\,(x, y) \end{aligned}\right\} \qquad \text{...(2a)}$$

$$\left.\begin{aligned} &u\,(x, 0, t) = 0,\ v\,(x, 0, t) = v_w\,(t) \\ &N\,(x, 0, t) = 0,\ T\,(x, 0, t) = T_w\,(t) \\ &u\,(x, \infty, t) = u_e\,(x, t),\ N\,(x, \infty, t) = 0,\ T\,(x, \infty, t) = T_\infty\,. \end{aligned}\right\} \qquad \text{...(2b)}$$

Here x and y are the distances along and perpendicular to the surface, respectively; t is the time; u and v are the velocity components in the x and y directions, respectively; N is the component of microrotation whose direction of rotations is in the $x - y$ plane; g is the acceleration due to gravity; ρ and T are the density and temperature of the fluid; μ, k_1 and γ are the viscosity, vortex viscosity and spin

gradient viscosity, respectively; β is the coefficient of thermal expansion; α_1 is the micropolar heat conduction coefficient; c_p is the specific heat of the fluid at constant pressure; j is the micro-inertia density; the subscripts t, x and y denote derivatives with respect to t, x and y, respectively; and the subscripts e and w denote conditions at the edge of the boundary layer and on the surface, respectively. The subscript i denotes values at the initial time $t = 0$ and T_∞ is a constant.

It may be remarked that we have assumed that the microrotation N is equal to zero on the boundary. The justification for using such a boundary condition is given in detail by Kirwan Jr[16]. Here the free-stream velocity u_e which vary with time can be expressed in the form

$$u_e = u_\infty \ \varphi(t^*), \ t^* = u_\infty \ t/L. \qquad \text{...(3)}$$

φ is an arbitrary function of the time t^* representing the nature of unsteadiness in the external stream and has a continuous first derivative for $t^* \geqslant 0$.

On applying the transformations

$$\left.\begin{aligned} &\eta = (u_\infty/2\nu\xi L)^{1/2}\, y,\ \xi = \bar{x},\ \bar{x} = x/L \\ &\psi = (2\nu u_\infty\ \xi\ L)^{1/2} f(\xi, \eta, t^*)\, \varphi(t^*) \end{aligned}\right\} \qquad \text{...(4a)}$$

$$\left.\begin{aligned} &u = u_e F,\ f' = F,\ u_e = u_\infty\ \varphi(t^*) \\ &v = -(\nu u_\infty/2\xi L)^{1/2}\, \varphi\, [f + 2\xi f_\xi - \eta F] \\ &N = [u_\infty^3\ 2\xi/(\nu L)]^{1/2}\, s \\ &(T - T_\infty)/(T_{w0} - T_\infty) = G \\ &(T_w - T_\infty)/(T_{w0} - T_\infty) = G\ \varphi_1(t^*) \end{aligned}\right\} \qquad \text{...(4b)}$$

$$\left.\begin{aligned} &f = \int_0^\eta F d\eta + f_w,\ f_w = A\ \xi^{1/2}/\varphi \\ &A = -(v_w/u_\infty)\,(\mathrm{Re}_L/2)^{1/2},\ \mathrm{Re}_L = u_\infty\ L/\nu \end{aligned}\right\} \qquad \text{...(4c)}$$

to (1), we find that (1a) is satisfied identically and (1b—d) reduce to

$$(1 + N_1)\, F'' + \varphi f F' + 2\xi\, [\varphi^{-1}\, \varphi_t^*\, (1 - F) - F_t^*] + 2\xi\, \varphi^{-1}\, [N_1 s' + \lambda G]$$
$$= 2\xi\, \varphi\, [F F_\xi - f_\xi\, F'] \qquad \text{...(5a)}$$

$$N_3\, s'' + N_2\, \varphi\, [f s' - s F] - N_1\, [4\xi\, s + \varphi\, F'] - 2\xi\, N_2\, s_t^*$$
$$= 2\xi\, \varphi\, N_2\, [F s_\xi - f_\xi\, s'] \qquad \text{...(5b)}$$

$$Pr^{-1}\, G'' + \varphi f G' + 2\xi\alpha\, [s\, G_\xi - s_\xi G'] - \alpha\, s G' + Br\, [1 + N_1/2]\, \varphi^2\, F'^2$$
$$+ 2 N_1 Br\, [2\xi\, s + \varphi\, F'/2]^2 + 2\xi\, N_3 Br s'^2 - 2\xi\, G_t^*$$
$$= 2\xi\, \varphi\, [F G_\xi - f_\xi\, G'] \qquad \text{...(5c)}$$

where

$$\left.\begin{aligned} &N_1 = k_1/\mu,\ N_2 = (u_\infty/L)\,(\rho\, j/\mu) \\ &N_3 = (u_\infty/L)\,(\gamma\rho/\mu^2),\ \alpha = (\alpha_1/\mu c_p)\,(u_\infty/L) \\ &\lambda = Gr/\mathrm{Re}_L^2\ ,\ Gr = g\beta\,(T_{w0} - T_\infty)\,L^3\,\rho^2/\mu^2 \\ &Br = u_\infty^2/[c_p\,(T_{w0} - T_\infty)]. \end{aligned}\right\} \quad \ldots(6)$$

Here ξ and η are transformed coordinates; t^* is the dimensionless time; u_∞ is the free-stream velocity at $t^* = 0$; L is the length of the plate; ν is the kinematic viscosity; ψ and f are the dimensional and dimensionless stream functions, respectively; $F(f')$, s and G are the dimensionless velocity, microrotation and temperature, respectively. The parameters N_1, N_2, N_3, α and Br are the coupling parameter, micro-intertia density parameter, microrotation parameter, micropolar heat conduction parameter and dissipation parameter, respectively; λ is the buoyancy parameter; Gr is the Grashof number; and Pr is the Prandtl number. f_w is the surface mass transfer parameter. If the normal velocity at the wall v_w is selected in such a manner that $(v_w/u_\infty)(\mathrm{Re}_L/2)^{1/2}$ is a constant then the parameter A will be a constant. Hence the mass transfer parameter f_w will vary according to (4c). $A \gtrless 0$ according to whether there is a suction or injection. The subscripts ξ and t^* denote derivatives with respect to ξ and t^*, respectively. The prime denotes derivatives with respect to η.

The transformed boundary conditions are given by

$$\left.\begin{aligned} &F = 0,\ s = 0,\ G = \varphi_1(t^*) \text{ at } \eta = 0 \\ &F \to 1,\ s \to 0,\ G \to 0 \qquad \text{as } \eta \to \infty \end{aligned}\right\} \text{ for } t^* \geqslant 0. \quad \ldots(7)$$

We assume that the flow is initially steady and then becomes unsteady for $t^* > 0$. Hence the initial conditions for F, s and G at $t^* = 0$ are given by steady flow equations obtained by putting

$$\varphi(t^*) = \varphi_1(t^*) = 1,\ \varphi_{t^*} = F_{t^*} = s_{t^*} = G_{t^*} = 0 \quad \ldots(8)$$

in (5) and also in (7) (for boundary conditions) and they are

$$(1 + N_1)\,F'' + fF' + 2\xi\,[N_1\, s' + \lambda\, G] = 2\xi\,[FF_\xi - f_\xi F'] \quad \ldots(9a)$$

$$N_3\, s'' + N_2\,[fs' - sF] - N_1\,[4\xi\, s + F'] = 2\xi\, N_2\,[Fs_\xi - f_\xi\, s'] \quad \ldots(9b)$$

$$\begin{aligned} Pr^{-1}\, G'' + fG' + 2\xi\alpha\,[s'G_\xi - s_\xi G'] - \alpha\, sG' + Br\,[1 + N_1/2]\,F'^2 \\ + 2N_1\, Br\,[2\xi s + F'/2]^2 + 2\xi\, N_3\, Br s'^2 = 2\xi\,[FG_\xi - f_\xi G']. \end{aligned} \quad \ldots(9c)$$

It may be remarked that the steady-state equations (9 (a)—(c)) reduce to that of Jena and Mathur[15] if we replace $2\xi\, N_1 s'$ by $N_1 s'$ in equation 9(a), $-N_2 sF$ by $N_2 sF$ and

$-N_1\,[4\xi s + F']$ by $-2\xi N_1\,[2s + F']$ in equation 9(b) and put $\alpha = Br = 0$ in equation 9(c). The equations 9(a)—(c) also reduce to those of mixed convection for Newtonian fluids which have been studied by Oosthuizen and Hart[3] and Gryzagoridis[4] if we put $N_1 = 0$ and replace $2\xi\lambda\, G$ by $\lambda\, G$ in 9(a), consequently, $s = 0$ and equation 9(b) becomes superfluous

The skin-friction coefficient at the wall is given by

$$C_f = 2\tau_w/[\rho\,(u_e^2)_{t^*=0}] = (2\,\mathrm{Re}_x^{-1})^{1/2}\,(1 + N_1)\,\varphi\,F'_w \qquad \text{...(10a)}$$

where

$$\tau_w = [(\mu + k_1)\,u_y + k_1\,N]_{y=0}. \qquad \text{...(10b)}$$

The heat-transfer coefficient in terms of Nusselt number is given by

$$Nu = 2xq_w/[k_c\,(T_{w0} - T_\infty)] = (2\,\mathrm{Re}_x)^{1/2}\,G'_w \qquad \text{...(11a)}$$

where

$$q_w = [k_c\,T_y + \beta_c\,N_x]_{y=0}. \qquad \text{...(11b)}$$

The couple stress coefficient is expressed in the form

$$M = m_w/[\rho\,(u_e^2)_{t^*=0}\,x] = (\mathrm{Re}_x)^{-1}\,N_3\,s'_w \qquad \text{...(12a)}$$

where

$$m_w = \gamma\,(N_y)_{y=0}. \qquad \text{...(12b)}$$

Here Cf, Nu and M are, respectively, the skin-friction coefficient, Nusselt number and couple stress coefficient; τ_w, q_w and m_w are, respectively, the shear stress, heat-transfer rate and couple stress at the wall; β_c is the heat conduction parameter and k_c is the thermal conductivity.

3. Results and Discussion

The equations (5a)—(5c) under boundary conditions (7) and initial conditions (9) have been solved numerically using an implicit finite-difference scheme with a quasi-linearization technique. Since the detailed description of the method is given in Bellman and Kalaba[17] and Inouye and Tate[18], its description is not repeated here. Computations have been a carried out for various values of the parameters $\lambda\,(-0.25 \leqslant \lambda \leqslant 20)$, $N_1\,(0.5 \leqslant N_1 \leqslant 13.5)$, $N_3\,(0.5 \leqslant N_3 \leqslant 4.5)$, $A\,(-0.5 \leqslant A \leqslant 0.5)$, $Pr\,(0.7 \leqslant Pr \leqslant 7.0)$ and $Br\,(-0.2 \leqslant Br \leqslant 0.2)$ with $N_2 = 1.0$, $\alpha = 1.0$, $\epsilon = 0.2$, $\epsilon_1 = 0.1$, $\epsilon_2 = 0.1$ and $\omega^* = 5.6$. The unsteady free-stream velocity and wall temperature distributions considered here are given by

$$\varphi\,(t^*) = 1 + \epsilon t^{*2},\ \varphi\,(t^*) = 1 + \epsilon_1 \sin^2\,(\omega^*\,t^*),\ \varphi_1\,(t^*) = 1 + \epsilon_2\,t^*$$

where ϵ, ϵ_1 and ϵ_2 are constants and ω^* is the frequency parameter. The effect of step sizes in η, ξ and t^* directions and the edge of the boundary layer η_∞ on the solution have been studied with a view to optimize them. Finally, the computations were carried out with $\Delta\eta = 0.02$, $\Delta\xi = 0.05$, $\Delta t^* = 0.1$ and η_∞ has been taken between 4 and 8 depending upon the values of the parameters. The results presented here found to be independent of the step size $\Delta\eta$, $\Delta\xi$, Δt^* and the edge of the boundary layer η_∞ at least up to 4th decimal place.

In order to asses the accurancy of method, we have compared our Nusselt number results for Newtonian fluids ($N_1 = 0$) with those of Oosthuizen and Hart[3] and Gryzagoridis[4]. We have also compared our skin-friction and heat-transfer results for micropolar fluids ($N_1 > 0$) for steady flow ($t^* = 0$) with those of Jena and Mathur[16]. In both the cases, the results are found to be in good agreement and the comparison is shown in Figs. 1 and 2.

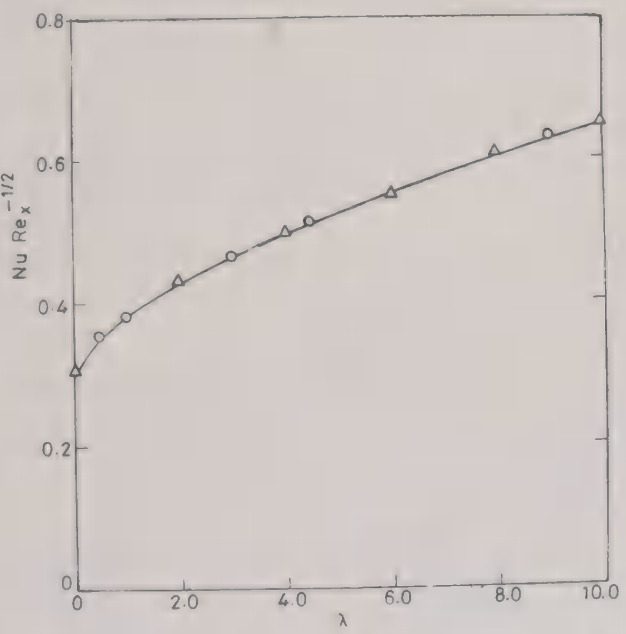

FIG. 1. Comparison of heat-transfer coefficient $Nu\,\mathrm{Re}_x^{-1/2}$ for $\varphi(t^*) = 1.0$, $\varphi_1(t^*) = 1.0$, $N_1 = N_2 = N_3 = \beta = A = Br = 0$. ———, present method; 0, Oosthuizen and Hart; Δ, Gryzagoridis.

The results for the case $\varphi(t^*) = 1 + \epsilon\, t^{*2}$, $\epsilon > 0$ (accelerating flow) are given in Figs. 3–6 and those for the case $\varphi(t^*) = 1 + \epsilon_1 \sin^2(\omega^* t^*)$ (fluctating flow) in Fig. 7.

The effect of buoyancy parameter λ, mass transfer parameter A, Prandtl number Pr, coupling parameter N_1, microrotation parameter N_3, variation of wall temperature with time $\varphi_1(t^*)$, distance ξ and dissipation parameter Br on the skin-friction, micro-rotation-gradient and heat-transfer parameters (F'_w, $-s'_w$, $-G'_w$) are shown in

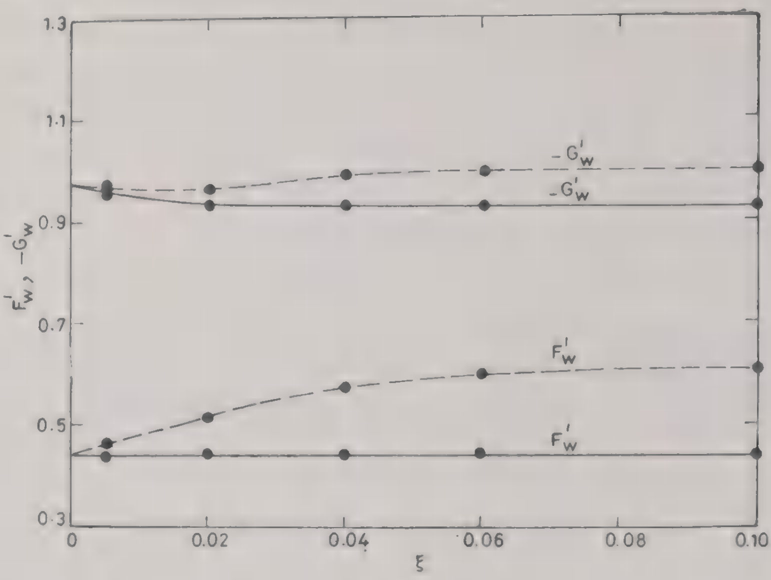

FIG. 2. Comparison of skin-friction parameter F'_w and heat-transfer parameter $-G'_w$ for φ (t^*) $=1.0$, φ_1 (t^*) $= 1.0$, $N_1 = 0.1$, $N_3 = 0.02$, $Pr = 9.0$, $A = 0.0$, $\alpha = 0.0$, $Br = 0.0$. ——, $\lambda = 0$; ——, $\lambda = 4$; ●, Jena and Mathur.

Figs. 3–5. Figures 6(a)—(c) depict the effect of buoyancy parameter λ on velocity, microrotation and temperature profiles.

The results indicate that the skin-friction parameter F'_w increases with time t^* for $\lambda \leqslant 0$ and decreases with it for $\lambda > 0$ whereas the microrotation-gradient parameter $-s'_w$ increases with t^* for values of λ [see Figs. 3(a)–(b)]. For $\lambda \leqslant 0$, the heat-transfer parameter $-G'_w$ increases with t^* but for $\lambda > 0$ it increases only after certain value of t^* [Fig. 3(c)]. It has also been observed that for all t^*, the parameters F'_w, $-s'_w$ and $-G'_w$ increase as λ increases. Similar behaviour has also been observed by Jena and Mathur[15] for steady case ($t^* = 0$). This is due to the fact that the buoyancy force ($\lambda > 0$) gives rise to favourable pressure gradient which accelerates the fluid in the boundary layer and thereby increases the skin-friction, microrotation-gradient and heat-transfer parameters. The skin-friction, microrotation-gradient and heat-transfer parameters (F'_w, $-s'_w$, $-G'_w$) are reduced due to injection ($A < 0$) whatever may be the value of t^* and the effect of suction ($A > 0$) is just the opposite. This is because injection increases the momentum, microrotation and thermal boundary layer thicknesses which cause deceleration in the fluid and suction does the opposite. Figures 3(a)–(b) also show that the skin-friction parameter F'_w and microrotation-gradient

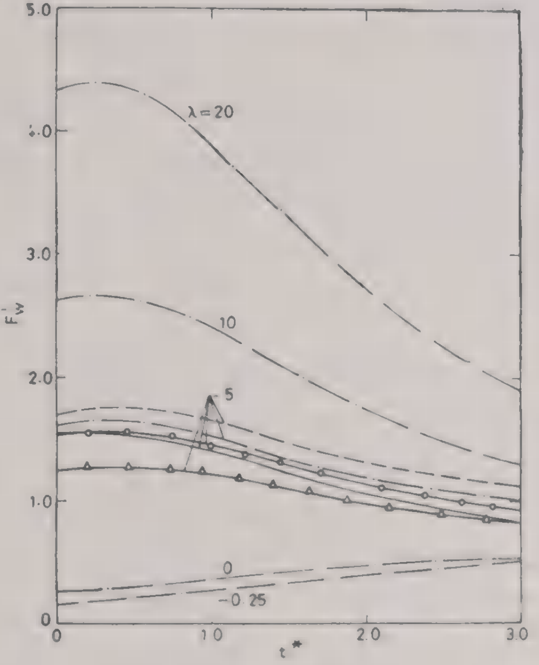

FIG. 3 (a) Skin-friction parameter F'_w

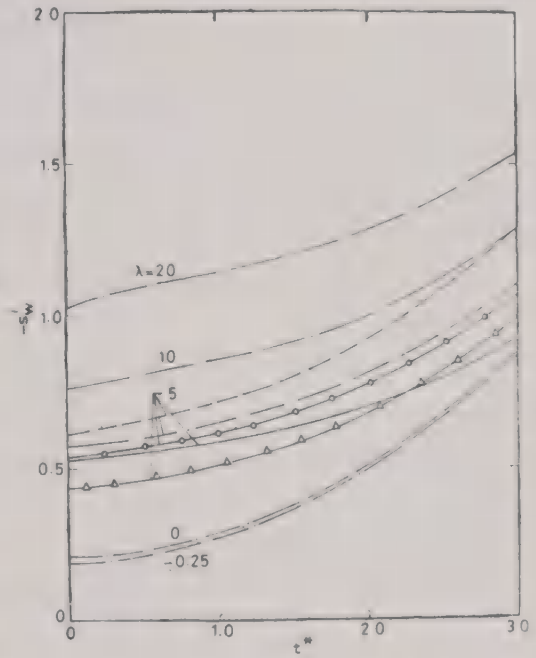

FIG. 3. (b) microrotation-gradient parameter $-s'_w$

FIG. 3 (c) Heat-transfer parameter $-G'_w$ for $\varphi(t^*) = 1 + \epsilon t^{*2}$, $\varphi_1(t^*) = 1.0$, $N_1 = 1.5$, $N_3 = 1.5$, $Br = 0.0$, $\xi = 0.5$. ——, $A = -0.5$, $Pr = 0.7$; ——.——, $A = 0.0$, $Pr = 0.7$; ———, $A = 0.5$, $Pr = 0.7$; ——○——, $A = 0.0$; $Pr = 3.0$——△——, $A = 0.0$, $Pr = 7.0$.

parameter $-s'_w$ decrease as Prandtl number Pr increases. Similar behaviour has also been observed by Wilks[2]. On the other hand, the heat-transfer parameter $-G'_w$ is found to increase with Pr (see Fig. 3(c)), because a large Prandtl number results in a thinner boundary layer with a corresponding large temperature at the wall and hence a large surface heat transfer. It has also been observed that the effect of Pr on the parameters F'_w and $-s'_w$ is less as compared to the parameter $-G'_w$.

Figures 4(a)–(c) show that any time t^*, the skin-friction, microrotation-gradient and heat-transfer parameters (F'_w, s'_w, $-G'_w$) decrease as the coupling parameter N_1 increases. Similar behaviour has also been observed by other investigators[10,19,20]. The cause of this reduction is due to the thickening of momentum, microrotation and thermal boundary layers due to in the parameter N_1 which in turn decelerates the fluid in the boundary layer. As N_3 increases, the parameters F'_w, $-s'_w$ and $-G'_w$ decrease whatever may be the value of t^*. It has also been observed from Figs. 4(a)–(c) that for $\xi = 0$, the parameters F'_w, $-s'_w$ and $-G'_w$ increase as time t^* increases from 0 to 3.0 but for $\xi > 0$, F'_w decreases with t^* and $-s'_w$ and $-G'_w$ increase with it. We have also observed that as ξ increases, the parameters F'_w and $-G'_w$ increase for $\lambda > 0$ and they decrease for $\lambda \leqslant 0$. Similar trend has been observed by Jena and Mathur[15] also. It is also clear from these figures that the parameters F'_w, $-s'_w$ and $-G'_w$ increase with the variation of the wall temperature with t^*. However, the heat-transfer parameter $-G'_w$ strongly affected by the variation of the wall temperature with t^* whereas its effect on F'_w and $-s'_w$ is rather weak.

The effect of dissipation parameter Br on the parameters F'_w, $-s'_w$ and $-G'_w$ is depicted in Fig. 5. This figure show that the parameters F'_w and $-s'_w$ increase as Br increases whereas the parameter $-G'_w$ decreases with it. This behaviour is independent of the value of t^*. For all values of Br, F'_w decreases with t^* (after certain t^*) whereas $-s'_w$ increases with it. As t^* increases, the parameter $-G'_w$ increases for $Br \leqslant 0$ and decreases for $Br > 0$. It has also beeen observed that the effect of Br is more pronounced on the parameter $-G'_w$ than on the parameters F'_w and $-s'_w$.

The effect of λ on the velocity, microrotation and temperature is shown in Figs. 6(a)–(c). Figure 6(a) shows that there is a velocity overshoot in F for buoyancy assisted

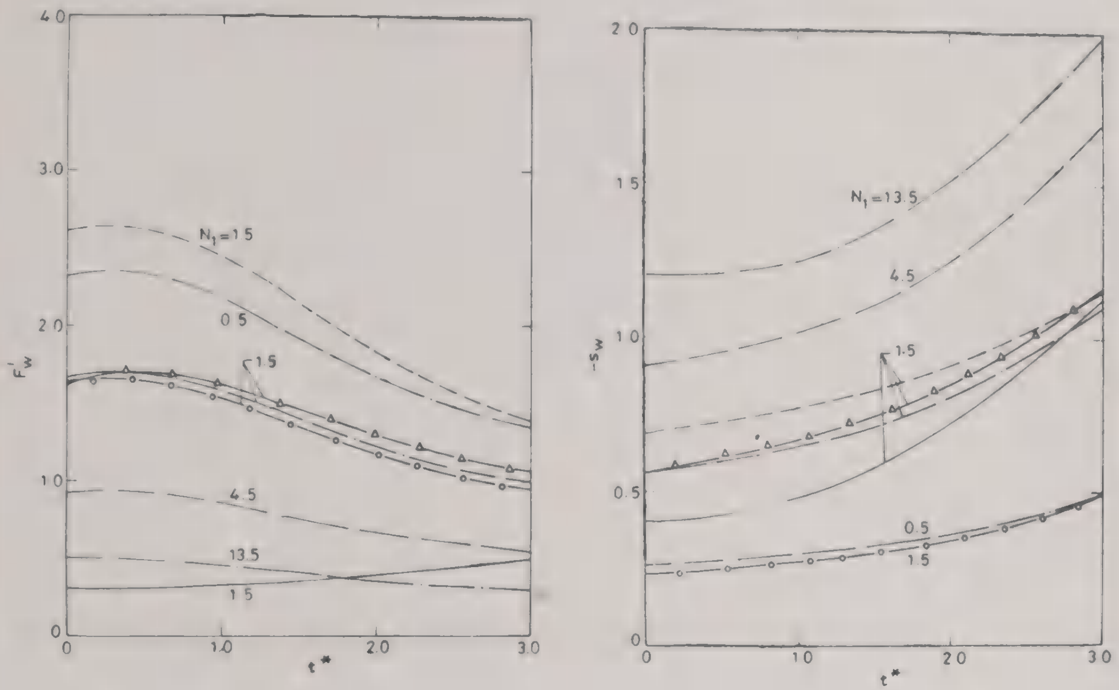

FIG. 4 (a) Skin-friction parameter F_w', (b) microrotation-gradient parameter $-s_w'$

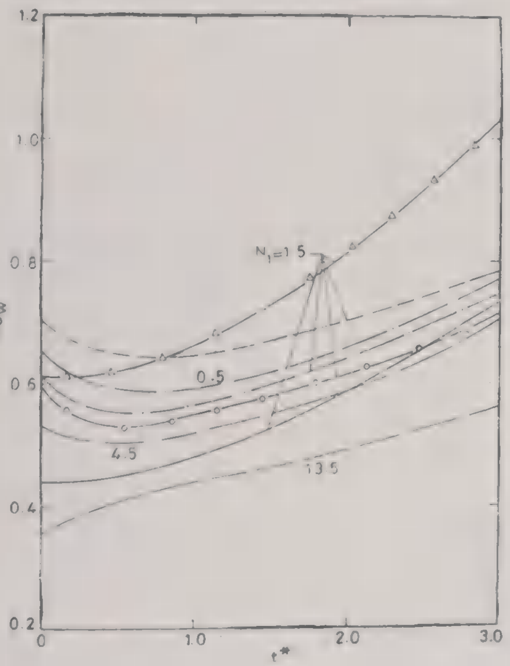

FIG. 4. (c) Heattransfer parameter $-G_w'$ for $\varphi(t^*) = 1 + \epsilon t^{*2}$, $Pr = 0.7$, $A = 0.0$, $\lambda = 5.0$, $Br = 0.0$.

———, $\xi = 0.0$, $N_3 = 1.5$, $\varphi_1(t^*)$ 1.0; ——.——, $\xi = 0.5$, $N_3 = 1.5$, $\varphi_1(t^*) = 1.0$; ———, $\xi = 1.0$, $N_3 = 1.5$, $\varphi_1(t^*) = 1.0$; ———○———, $\xi = 0.5$, $N_3 = 4.5$, $\varphi_1(t^*) = 1.0$; ———△——–, $\xi = 0.5$, $N_3 = 1.5$, $\varphi_1(t^*) = 1 + \epsilon_2 t^*$.

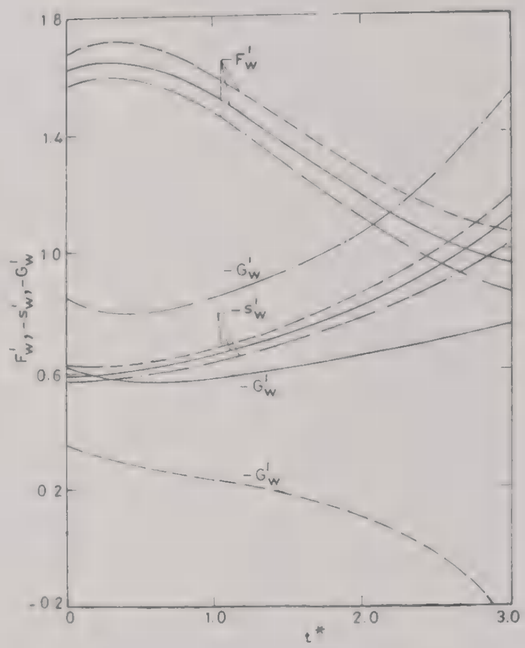

FIG. 5. Skin-friction parameter F'_w, microrotation-gradient parameter $-s'_w$ and heat-transfer parameter $-G'_w$ for $\varphi(t^*) = 1 + \epsilon t^{*2}$, $\varphi_1(t^*) = 1.0$, $N_1 = 1.5$, $N_3 = 1.5$, $A = 0.0$, $\lambda = 5.0$, $Pr = 0.7$, $\xi = 0.5$. ———, $Br = 0.0$; ——.——, $Br = -0.2$; ———, $Br = 0.2$.

flow ($\lambda > 0$), and the velocity overshoot increses as λ increases. However it decreases as t^* increases. There is no velocity overshoot either for purely forced flow ($\lambda = 0$) or buoyancy opposed flow ($\lambda < 0$). The velocity overshoot is because buoyancy force ($\lambda > 0$) gives rise to a favourable pressure gradient resulting in velocity which adds to the forced convection velocity. The buoyancy opposed flow ($\lambda < 0$) gives to adverse

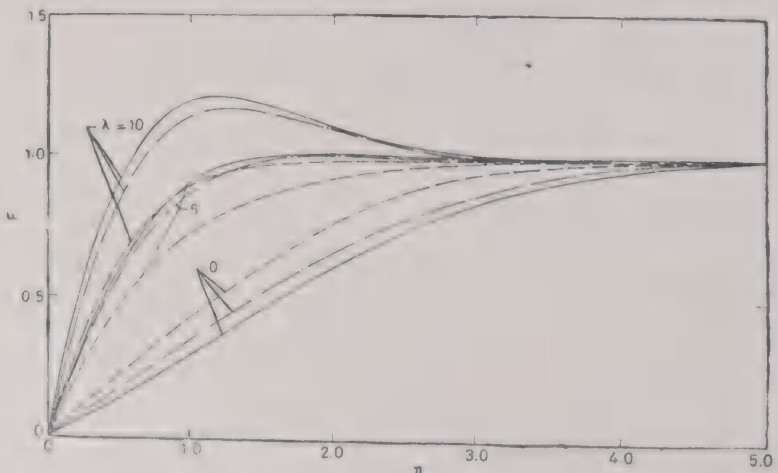

FIG. 6. (a) Velocity profile in the x direction F.

pressure gradient which reduces the forced convection velocity. As the combined effect of buoyancy force $\lambda > 0$ and forced convection force decrease with time which results in reduction in velocity overshoot with time.

The microrotation profile $-s$ and the temperature profile G are shown in Figs. 6(b)–(c). It is observed that the profiles $-s$ and G are significantly affected by the parameter λ and the effect becomes more pronounced as time t^* increases. The profiles $-s$ and G become more steep as t^* and λ increase.

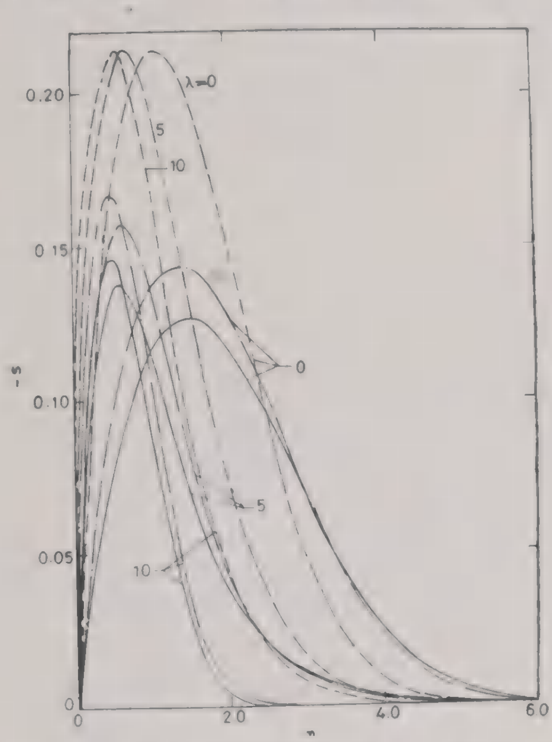

FIG. 6. (b) microrotation profile $-s$.

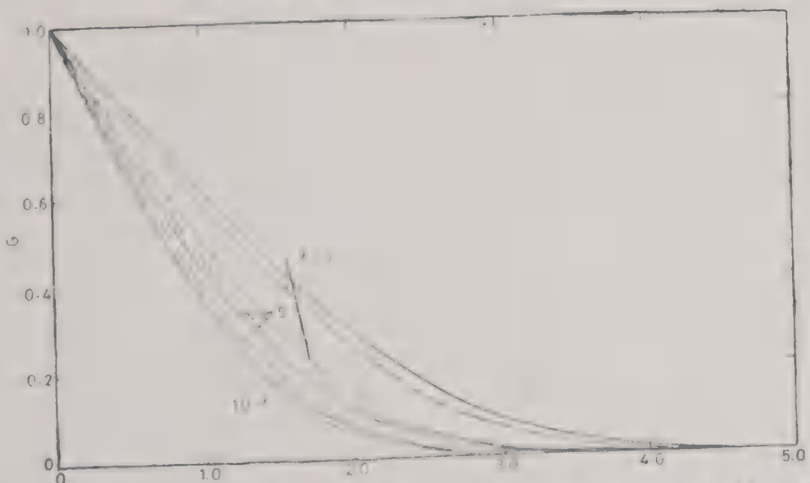

FIG 6. (c) Temperature profile G for $\varphi(t^*) = 1 + \epsilon t^{*2}$, $\varphi_1(t^*) = 1.0$; $N_1 = 1.5$, $N_3 = 1.5$, $Pr = 0.7$, $A = 0.0$, $Br = 0.0$, $\xi = 0.5$, ––––, $t^* = 0.0$; –––.–––, $t^* = 1.0$; ––––––, $t^* = 2.0$.

The skin-friction, microrotation-gradient and heat-transfer parameters (F_w', $-s_w'$, $-G_w'$) for oscillatory free-stream velocity $\varphi(t^*) = 1 + \epsilon_1 \sin^2(\omega^* t^*)$ are shown in

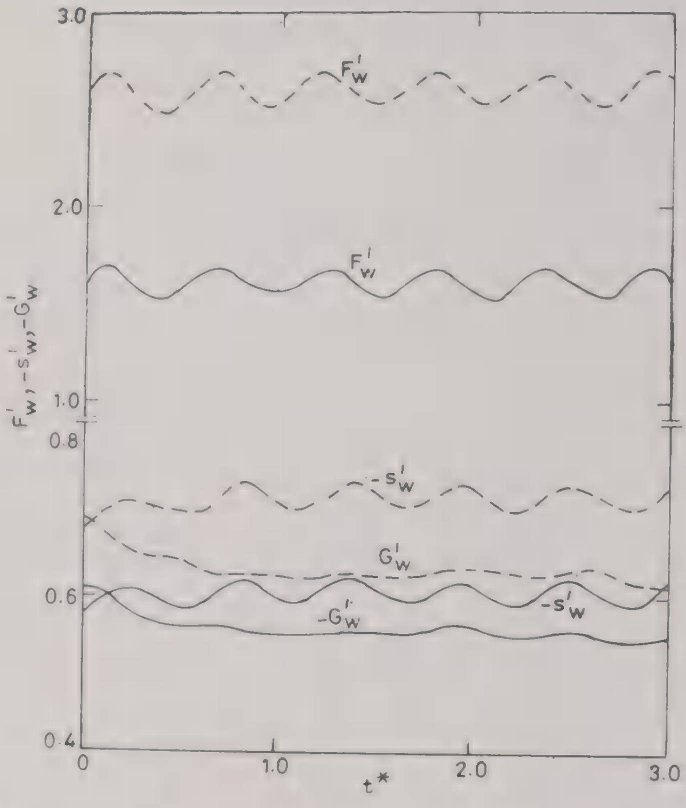

FIG. 7. Skin-friction parameter F_w', microrotation-gradient parameter $-s_w'$ and heat-transfer parameter $-G_w'$ for $\varphi(t^*) = 1 + \epsilon_1 \sin^2(\omega^* t^*)$, $\varphi_1(t^*) = 1.0$, $N_1 = 1.5$, $N_3 = 1.5$, $Pr = 0.7$, $A = 0.0$, $\lambda = 5.0$, $Br = 0.0$. ————, $\xi = 0.5$; — — — $\xi = 1.0$.

Fig. 7. It is clear from this figure that the parameters F_w', $-s_w'$ and $-G_w'$ oscillate as time t^* increases but the oscillations are more for large ξ.

4. Conclusions

The skin friction, microrotation gradient and heat transfer are strongly dependent on the buoyancy parameter, coupling parameter, mass transfer and unsteadiness in free-stream velocity. The effect of microrotation parameter on microrotation gradient is appreciable whereas its effect on skin friction and heat transfer is comparatively small. The Prandtl number, dissipation parameter and the variation of the wall temperature with time affect the heat transfer significantly whereas skin friction and microrotation gradient are weakly affected by it. Buoyancy parameter induces over-

shoot in the velocity profiles. The magnitude of the velocity overshoot increases as the buoyancy parameter increases and it decreases as time increases.

Acknowledgement

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PROPAGATION OF SHOCK WAVES IN A NONHOMOGENEOUS ELASTIC MEDIUM WITH A SPHERICAL HOLE

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The propagation of elastic disturbance in a nonhomogeneous medium due to radial pressure at the boundary of a spherical cavity has been studied. The problem admits of a closed form solution in some special cases only. The cases where no closed form solution is available have been investigated following Kromm's technique. The solution has been obtained in terms of integral equation. Distribution of stresses have been shown graphically.

1. Introduction

The problem of the propagation of elastic disturbance in an infinite medium caused by a radial shock is the subject of interest of many investigators for its importance in many practical fields viz. in seismology, in geophysical prospecting and in many other fields as has been discussed in detail by Neuber and Hahn[1]. We may think of a situation where there is spherical hole in a very large elastic medium and an explosion within the hole leads to release of huge amount of gas pressing the spherical boundary of the medium radially. Our problem is to study the propagation of elastic disturbance due to this radial pressure. The problem of this type for a homogeneous isotropic medium was apparently solved first by Jeffreys[2]. But as we know that all elastic media cannot be considered homogeneous and as it has been experimentally verified that there are plenty of media in which the elastic coefficients are functions of position, it is our motivation to study the problem for a more realistic medium, viz. a medium where the elastic co-efficients are not merely constants but are functions of position. The problem for a nonhomogeneous medium had also been considered by a number of investigators[3–5] but in all those papers some restrictions were imposed to get the solution.

In our present investigation we shall study the problem similar to those discussed above but with a modification that the boundary of the cavity is disturbed suddenly

and thereafter maintained steadily, the cavity being free from any other type of loading. With such a modification it is interesting to note that it has been possible to obtain complete solution of the problem for all values of the parameters involved. In our method of solution we have used the techniques followed by Kromm[6] and Goodier and Jahsman[7]. The expressions for stresses and displacement have been presented in the form of integral equations, which may be solved numerically. The expressions for the velocity of the wave front and also the values of the stresses and displacement at the wave front have been given. Finally the stress distributions for some particular cases have been shown graphically.

2. Formulation of the Problem

Let the centre of the spherical cavity be taken as the pole of the spherical polar co-ordinate system (r, θ, φ). In view of symmetry, the displacement components $(u_r, u_\theta, u_\varphi)$, may be written as

$$u_r = u\,(r, t),\; u_\theta = u_\varphi = 0. \qquad \ldots(1)$$

The stress components τ_{rr}, $\tau_{r\theta}$ etc. are connected with the displacement components (1) as,

$$\begin{aligned} \tau_{rr} &= \frac{E}{(1+\sigma)(1-2\sigma)}\left[(1-\sigma)\frac{\partial u}{\partial r} + 2\sigma\frac{u}{r}\right] \\ \tau_{\theta\theta} = \tau_{\varphi\varphi} &= \frac{E}{(1+\sigma)(1-2\sigma)}\left[\sigma\frac{\partial u}{\partial r} + \frac{u}{r}\right] \\ \tau_{\theta\varphi} = \tau_{r\theta} &= \tau_{r\varphi} = 0,\; 0 < \sigma < \tfrac{1}{2}. \end{aligned} \qquad \ldots(2)$$

In (2) E is the variable Young's modulus and σ the constant Poisson's ratio of the material.

In view of (2) the equation of motion becomes

$$\frac{\partial}{\partial r}\tau_{rr} + \frac{2}{r}(\tau_{rr} - \tau_{\theta\theta}) = \rho\frac{\partial^2 u}{\partial t^2}. \qquad \ldots(3)$$

ρ being the constant mass density.

We assume that the Young's modulus E depends upon the position according to the law,

$$E = E_0\,(r/a)^n \qquad \ldots(4)$$

where parameter n may be any real number.

Introducing nondimensional variables

$$z = r/a \text{ and } \tau = ct/a \text{ where } c^2 = E_0/\rho \qquad \ldots(5)$$

and taking into account of (2) and (4), the field equation (3) becomes,

$$\frac{\partial^2 u}{\partial z^2} + (n+2)\frac{1}{z}\frac{\partial u}{\partial z} + \left(\frac{2n\sigma}{1-\sigma} - 2\right)\frac{u}{z^2} = \frac{\lambda^2}{z^n}\frac{\partial^2 u}{\partial \tau^2} \qquad \text{...(6)}$$

in which

$$\lambda^2 = \frac{(1+\sigma)(1-2\sigma)}{1-\sigma}.$$

Assuming that at $\tau = 0$ there is neither displacement nor velocity in the medium the initial conditions of the problem may be put as,

$$u(z, 0) = 0$$

$$\frac{\partial}{\partial \tau} u(z, 0) = 0, \; 1 < z < \infty. \qquad \text{...(7)}$$

As regards the boundary condition we assume that the boundary $z = 1$ is disturbed and thereafter the disturbance is steadily maintained. This may be expressed mathematically as,

$$\tau_{rr}(1, \tau) = -PH(\tau), \; \tau > 0 \qquad \text{...(8)}$$

where P is a constant and $H(\tau)$ is the Heaviside function.

In addition to (8) the regularity condition requiring boundedness of the stresses and displacement for all values of z and τ should be satisfied.

Thus the solution of the problem reduces to the solution of (6) satisfying initial conditions (7), boundary condition (8) and also the regularity condition.

3. Solution of the Problem

In view of (7), Laplace transforms with respect to τ of the field equation (6) and the boundary condition (8) yield

$$\frac{d^2\bar{u}}{dz^2} + (n+2)\frac{1}{z}\frac{d\bar{u}}{dz} + \left(\frac{2n\sigma}{1-\sigma} - 2\right)\frac{\bar{u}}{z^2} = \frac{\lambda^2}{z^n} p^2\bar{u} \qquad \text{...(9)}$$

and

$$\bar{\tau}_{rr} = -\frac{P}{p} \text{ at } z = 1 \qquad \text{...(10)}$$

where the bar over a function represents Laplace transform of the function with respect to parameter p.

Since the character of the solution of (9) depends on whether $n < 2$, $n = 2$ or $n > 2$, we consider three cases, viz., $n < 2$, $n = 2$ and $n > 2$.

For $n \neq 2$, the general solution of (9) is,

$$\bar{u} = Az^{\beta} K_m(pkz^{\alpha}) + Bz^{\beta} I_m(pkz^{\alpha}) \qquad \text{...(11)}$$

where

$$\alpha = 1 - (n/2),\ \beta = -(n+1)/2,\ k = \lambda/\alpha$$

and

$$m^2 = \frac{1}{\alpha^2}\left[\frac{(n+1)^2}{4} + 2 - \frac{2n\sigma}{1-\sigma}\right] > 0 \text{ as } 0 < \sigma < \tfrac{1}{2}.$$

For $n < 2$, the argument of the Bessel functions in (11) is positive for $p > 0$ and tends to infinity as $z \to \infty$. In this case the solution of (9) satisfying the regularity requirements is,

$$\bar{u} = A z^{\beta} K_m(pkz^{\alpha}). \qquad \ldots(12)$$

The arbitrary constant $A = A(p)$ in (12) may be determined from (10). Finally we obtain,

$$\bar{u} = -\frac{2Pa(1+\sigma)(1-2\sigma)}{DE_0 p} z^{-(n+1)/2} K_m(pkz^{\alpha}) \qquad \ldots(13)$$

$$\bar{\tau}_{rr} = -\frac{Pz^{(n-3)/2}}{Dp}[(1-\sigma)(2-n)\, pkz^{\alpha} K'_m(pkz^{\alpha})$$
$$+ \{4\sigma - (1-\sigma)(n+1)\} K_m(pkz^{\alpha})] \qquad \ldots(14)$$

$$\bar{\tau}_{\theta\theta} = -\frac{Pz^{(n-3)/2}}{Dp}[\sigma(2-n)\, pkz^{\alpha} K'_m(pkz^{\alpha})$$
$$+ \{2 - (n+1)\sigma\} K_m(pkz^{\alpha})] \qquad \ldots(15)$$

where

$$D = (1-\sigma)(2-n)\, pk\, K'_m(pk) + \{4\sigma - (1-\sigma)(n+1)\} K_m(pk).$$

Introducing the notations,

$$a_1 = (1-\sigma)(2-n)$$

$$b_1 = 4\sigma - (1-\sigma)(n+1)$$

$$a_2 = \sigma(2-n)$$

$$b_2 = 2 - (n+1)\sigma$$

$$\theta(z) = -\frac{2Pa}{E_0}(1+\sigma)(1-2\sigma) z^{-(n+1)/2}$$

$$\Psi(z) = -P z^{\frac{n-3}{2}}$$

and using properties of Bessel function (*cf.* Abramowitz and Stegun[8]) we may rewrite (13), (14) and (15) as,

$$\bar{u}(z, p) = \frac{\theta(z)}{p} \frac{K_m(pkz^\alpha)}{\Delta(p)} \quad \ldots(16)$$

$$\bar{\tau}_{rr}(z, p) = \frac{\psi(z)}{p} \frac{\Delta_1(p)}{\Delta(p)} \quad \ldots(17)$$

$$\bar{\tau}_{\theta\theta}(z, p) = \frac{\psi(z)}{p} \frac{\Delta_2(p)}{\Delta(p)} \quad \ldots(18)$$

where

$$\Delta(p) = (a_1 m + b_1) K_m(pk) - a_1 pk K_{m+1}(pk)$$

$$\Delta_i(p) = (a_i m + b_i) K_m(pkz^\alpha) - a_i pkz^\alpha K_{m+1}(pkz^\alpha) \; (i = 1, 2).$$

Let us set,

$$\left.\begin{aligned}
\bar{\varphi}(p) &= (a_1 m + b_1) \frac{K_m(pk)}{p} - a_1 k K_{m+1}(pk) \\
\bar{\psi}_1(z, p) &= \frac{\theta(z)}{p} \frac{K_m(pkz^\alpha)}{p} \\
\bar{\psi}_2(z, p) &= \frac{\psi(z)}{p} \left\{(a_1 m + b_1) \frac{K_m(pkz^\alpha)}{p} - a_1 kz^\alpha K_{m+1}(pkz^\alpha)\right\} \\
\bar{\psi}_3(z, p) &= \frac{\psi(z)}{p} \left\{(a_2 m + b_2) \frac{K_m(pkz^\alpha)}{p} - a_2 kz^\alpha K_{m+1}(pkz^\alpha)\right\}.
\end{aligned}\right\} \quad \ldots(19)$$

Then we may get (Erdelyi[9])

$$\varphi(\tau) = \frac{H(\tau - k)}{2} \left[\frac{a_1 m + b_1}{m} \left\{(\omega_1 + \sqrt{\omega_1^2 - 1})^m - (\omega_1 - \sqrt{\omega_1^2 - 1})^m\right\}\right.$$
$$\left. - a_1 k (\tau^2 - k^2)^{-1/2} \{(\omega_1 + \sqrt{\omega_1^2 - 1})^{m+1} + (\omega_1 - \sqrt{\omega_1^2 - 1})^{m+1}\}\right]$$

$$\psi_1(\tau) = \theta(z) \int_0^\tau \frac{H(\xi - k_1)}{2m} \left[(\omega + \sqrt{\omega^2 - 1})^m - (\omega - \sqrt{\omega^2 - 1})^m\right] d\xi$$

$$\psi_2(\tau) = \psi(z) \int_0^\tau \frac{H(\xi - k_1)}{2} \left[\frac{a_1 m + b_1}{m}\right.$$
$$\left\{(\omega + \sqrt{\omega^2 - 1})^m - (\omega - \sqrt{\omega^2 - 1})^m\right\}$$
$$- a_1 k_1 (\xi^2 - k_1^2)^{-1/2} \left\{(\omega + \sqrt{\omega^2 - 1})^{m+1}\right.$$
$$\left.\left. + (\omega - \sqrt{\omega^2 - 1})^{m+1}\right\}\right] d\xi$$

$$\psi_3(\tau) = \psi(z) \int_0^{\tau} \frac{H(\xi - k_1)}{2} \Big[\frac{a_2 m + b_2}{m}$$

$$\Big\{ (\omega + \sqrt{\omega^2 - 1})^m - (\omega - \sqrt{\omega^2 - 1})^m \Big\}$$

$$- a_2 k_1 (\xi^2 - k_1^2)^{-1/2} \Big\{ (\omega + \sqrt{\omega^2 - 1})^{m+1}$$

$$+ (\omega - \sqrt{\omega^2 - 1})^{m+1} \Big\} \Big] d\xi \qquad \ldots (20)$$

where

$$\omega_1 = \tau/k,\ \omega = \xi/k_1 \text{ and } k_1 = kz^{\alpha}.$$

From (16), (17), (18) and (19) we have

$$\bar{u}(z, p)\, \bar{\varphi}(p) = \bar{\psi}_1(z, p) \qquad \ldots (21a)$$

$$\bar{\tau}_{rr}(z, p)\, \bar{\varphi}(p) = \bar{\psi}_2(z, p) \qquad \ldots (21b)$$

$$\bar{\tau}_{\theta\theta}(z, p)\, \bar{\varphi}(p) = \bar{\psi}_3(z, p). \qquad \ldots (21c)$$

Now taking inverse Laplace transform of (21a, b, c) we get,

$$\int_0^{\tau} u(z, \xi)\, \varphi(\tau - \xi)\, d\xi = \psi_1(z, \tau) \qquad \ldots (22a)$$

$$\int_0^{\tau} \tau_{rr}(z, \xi)\, \varphi(\tau - \xi)\, d\xi = \psi_2(z, \tau) \qquad \ldots (22b)$$

$$\int_0^{\tau} \tau_{\theta\theta}(z, \xi)\, \varphi(\tau - \xi)\, d\xi = \psi_3(z, \tau). \qquad \ldots (22c)$$

The integral equations (22a, b, c) may be reduced to comparatively simple forms suitable for numerical evaluation. Thus introducing the notations,

$$\varphi_1(\tau) = \frac{a_1 m + b_1}{2m} \Big\{ (\omega_1 + \sqrt{\omega_1^2 - 1})^m - (\omega_1 - \sqrt{\omega_1^2 - 1})^m \Big\}$$

$$- \frac{a_1 k (\tau^2 - k^2)^{-1/2}}{2} \Big\{ (\omega_1 + \sqrt{\omega_1^2 - 1})^{m+1} + (\omega_1 - \sqrt{\omega_1^2 - 1})^{m+1} \Big\}$$

$$f_1(z, \tau) = \int_0^{\tau - k_1} \frac{1}{2m} \Big[(\omega + \sqrt{\omega^2 - 1})^m - (\omega - \sqrt{\omega^2 - 1})^m \Big] d\zeta$$

$$f_2(z,\tau) = \int_0^{r-k_1} \left[\frac{a_1 m + b_1}{2m} \left\{ (\omega + \sqrt{\omega^2 - 1})^m (\omega - \sqrt{\omega^2 - 1})^m \right\} \right.$$

$$- a_1 k_1 (\zeta^2 + 2\zeta k_1)^{-1/2} \tfrac{1}{2} \left\{ (\omega + \sqrt{\omega^2 - 1})^{m+1} \right.$$

$$\left. \left. + (\omega - \sqrt{\omega^2 - 1})^{m+1} \right\} \right] d\zeta$$

$$f_3(z,\tau) = \int_0^{r-k_1} \left[\frac{a_2 m + b_2}{2m} \left\{ (\omega + \sqrt{\omega^2 - 1})^m - (\omega - \sqrt{\omega^2 - 1})^m \right\} \right.$$

$$- a_2 k_1 (\zeta^2 + 2\zeta k_1)^{-1/2} \tfrac{1}{2} \left\{ (\omega + \sqrt{\omega^2 - 1})^{m+1} \right.$$

$$\left. \left. + (\omega - \sqrt{\omega^2 - 1})^{m+1} \right\} \right] d\zeta$$

$$(\zeta = \xi - k_1)$$

we obtain,

$$\varphi(\tau) = H(\tau - k)\,\varphi_1(\tau)$$

$$\psi_1(z,\tau) = \theta(z)\,H(\tau - k_1)\,f_1(z,\tau)$$

$$\psi_2(z,\tau) = \psi(z)\,H(\tau - k_1)\,f_2(z,\tau)$$

$$\psi_3(z,\tau) = \psi(z)\,H(\tau - k_1)\,f_3(z,\tau)$$

and eqns. (22a, b, c) take the form,

$$\int_0^{\tau_1} u(z,\xi)\,\varphi_1(\tau_1 + k - \xi)\,d\xi = H(\tau_1 - \delta_1)\,\theta(z)\,f_1(z, \tau_1 + k) \quad \ldots(23a)$$

$$\int_0^{\tau_1} \tau_{rr}(z,\xi)\,\varphi_1(\tau_1 + k - \xi)\,d\xi = H(\tau_1 - \delta_1)\,\psi(z)\,f_2(z, \tau_1 + k) \quad \ldots(23b)$$

$$\int_0^{\tau_1} \tau_{\theta\theta}(z,\xi)\,\varphi_1(\tau_1 + k - \xi)\,d\xi = H(\tau_1 - \delta_1)\,\psi(z)\,f_3(z, \tau_1 + k) \quad \ldots(23c)$$

where

$\tau_1 = \tau - k$ and $\delta_1 = k_1 - k$.

The integral equations (23b, c) are to be numerically solved to obtain the stresses $\tau_{rr}(z,\tau)$ and $\tau_{\theta\theta}(z,\tau)$. We shall follow the usual procedure of replacing the integral equation with approximately equivalent finite system of linear algebric equations, assuming that the unknown function is piecewise constant over the range of integration.

Since the right-hand members of the equations (23b, c) contain $H(\tau_1 - \delta_1)$ as factor it follows that $\tau_{rr}(z, \tau_1) = 0$ and $\tau_{\theta\theta}(z, \tau_1) = 0$ for $\tau_1 < \delta_1$ and as $\delta_1 = k(z^{\alpha} - 1)$, $(k > 0, \alpha > 0)$ it follows that $\delta \to \infty$ as $z \to \infty$. This shows that the stress waves progress towards infinity in infinite time. If z and τ momentarily, denote instantaneous radius of the wave front and the time at which this radius is attained, then the velocity of the propagation of the wave front is given by

$$v = \frac{dz}{d\tau} = \frac{1}{\lambda} z^{n/2}.$$

Consequently velocity increases or decreases with z according as $n > 0$ or < 0.

The values of the displacement and stresses at the wave front may be obtained as,

$$u(z, \delta_1 +) = \lim_{\epsilon \to 0} \frac{\theta(z) f_2(z, k_1 + \epsilon)}{\Pi} = 0$$

$$\tau_{rr}(z, \delta_1 +) = \lim_{\epsilon \to 0} \frac{\psi(z) f_2(z, k_1 + \epsilon)}{\Pi} = \psi(z) z^{\alpha/2}$$

$$\tau_{\theta\theta}(z, \delta_1 +) = \lim_{\epsilon \to 0} \frac{\psi(z) f_3(z, k_1 + \epsilon)}{\Pi} = \psi(z) \frac{a_2}{a_1} z^{\alpha/2}$$

where

$$\Pi = \int_{\delta_1}^{\delta_1 + \epsilon} \varphi_1(\delta_1 + \epsilon + k - \xi)\, d\xi.$$

For $n = 2$, solution of (9) is given by,

$$\bar{u} = A_1 z^{m_1} + B_1 z^{m_2} \quad \ldots(24)$$

where

$$m_1, m_2 = \tfrac{1}{2}(-3 \pm \sqrt{9 + 4\epsilon_1})$$

and

$$\epsilon_1 = \lambda^2 p^2 + 2 - 4\sigma/(1 - \sigma). \quad \ldots(25)$$

We shall assume that the parameter p in Laplace transform is such that $\epsilon_1 > 0$ in (25). So the appropriate solution will be $\bar{u} = B_1 z^{m_2}$.

Applying the boundary condition (10) we can find B_1. Finally the inversion formula of Laplace transform gives (Erdelyi[9])

$$u = -\frac{Pa(1 + \sigma)(1 - 2\sigma)}{\lambda E_0 (1 - \sigma)} z^{-3/2} \int_0^{\tau} g_1(\xi)\, d\xi$$

where

$$g_1(\tau) = f(\tau) - \nu \int_0^\tau f(\sqrt{\tau^2 - v^2})\, J_1(\nu v)\, dv$$

$$f(\tau) = e^{-(\tau-y)\eta},\ \tau > y$$

$$= 0 \qquad ,0 < \tau < y$$

$$y = \log z\ (z > 1)$$

$$\eta = (3 - 7\sigma)/2\lambda\,(1 - \sigma)$$

$$\nu^2 = (17 - 33\sigma)/4\lambda^2\,(1 - \sigma)$$

and J_1 is the Bessel function of the first kind and of order 1.

The components of stresses are then computed as,

$$\tau_{rr} = -P\, z^{-1/2} \int_0^\tau g_2(\xi)\, d\xi$$

$$\tau_{\theta\theta} = -P\, z^{-1/2} \int_0^\tau g_3(\xi)\, d\xi$$

where

$$g_2(\tau) = \delta(\tau - y) - \nu \int_0^\tau \delta(\sqrt{\tau^2 - v^2} - y)\, J_1(\nu v)\, dv$$

$$g_3(\tau) = g(\tau) - \nu \int_0^\tau g(\sqrt{\tau^2 - v^2} - y)\, J_1(\nu v)\, dv$$

$$g(\tau) = \frac{\sigma}{1 - \sigma}\,\delta(\tau - y) - \frac{(1 + 2\sigma)(1 + \sigma)}{\lambda(1 - \sigma)^2}\, e^{-\eta(\tau-y)},\ \tau > y$$

$$= 0,\ 0 < \tau < y$$

$\delta(\tau)$ being the Dirac delta function.

For $n > 2$, we find that $\alpha < 0$ and therefore the argument of the Bessel functions is negative for $p > 0$, and then neither I_m nor K_m is real valued. Moreover, the argument now tends to zero as $z \to \infty$. Since at $x = 0$, $I_m(x)$ and $K_m(x)$ have linearly independent singularities, the general solution (11) cannot satisfy regularity requirement. Thus we are compelled to abandon the regularity condition. This brings us to the difficulty of unique determination of two constants A and B in (11) with the help of single boundary condition (10). However, it has been pointed out by Sternberg and Chakravorty[10] that for the possibility of a diverging wave in a physical domain B must be equal to zero and so the appropriate solution, in this case also,

will be given by (12). Thus proceeding as before we obtain the same expression for the displacement and stresses as in the case $n < 2$.

4. Numerical Results

In order to get some idea about the behaviour of the stresses with time at different positions of the medium for different values of the exponent we have computed the values of $R = -\dfrac{\tau_{rr}}{P}$ and $S = -\dfrac{\tau_{\theta\theta}}{P}$ for different values of z, τ_1 and n corresponding to $\sigma = 1/3$. In Fig. 1 and Fig. 3, the value of the nonhomogeneity

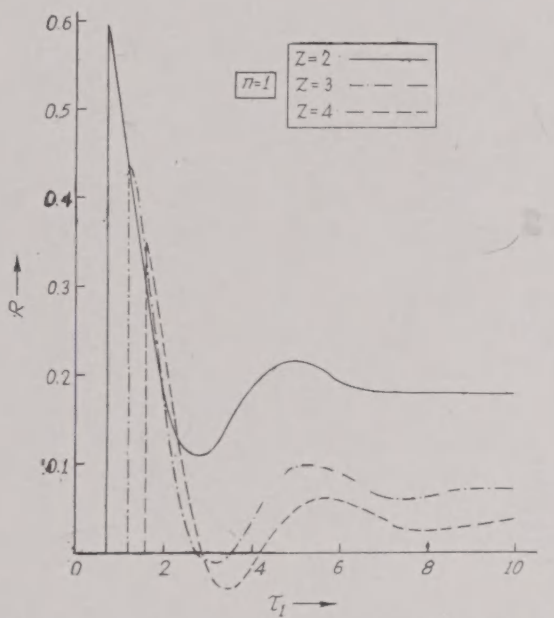

Fig. 1. Variation of τ_{rr} with time (for fixed η).

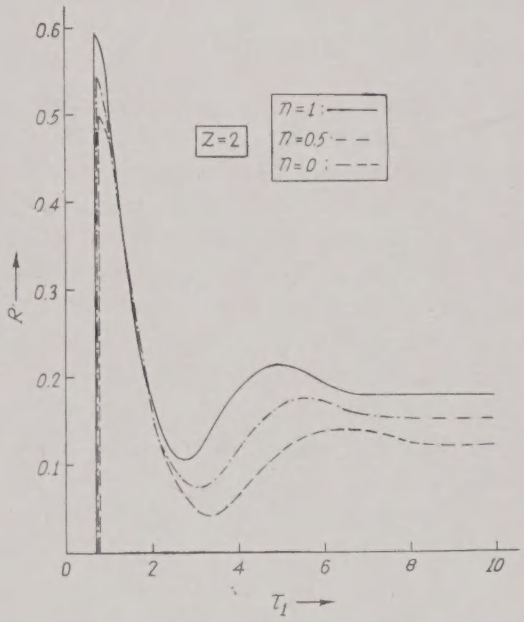

Fig. 2. Variation of τ_{rr} with time (for fixed position).

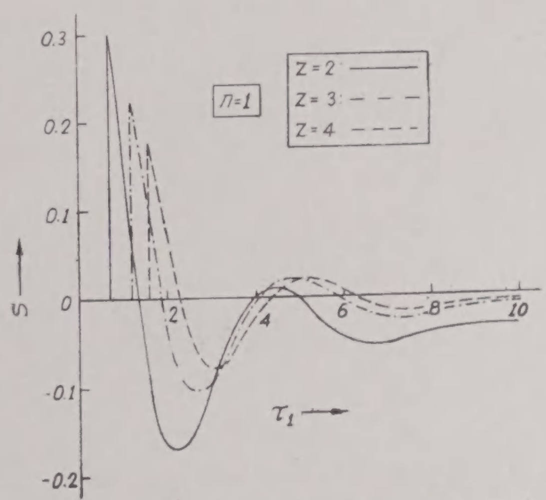

Fig. 3. Variation of $\tau_{\theta\theta}$ with time (for fixed η).

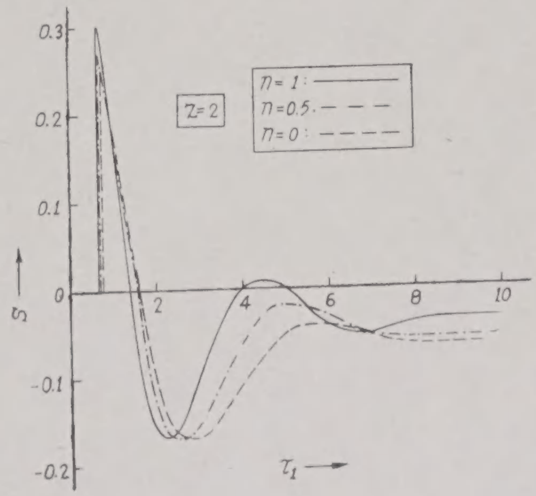

Fig. 4. Variation of $\tau_{\theta\theta}$ with time (for fixed position).

parameter n has been kept fixed at 1 and values of R and S have been evaluated respectively for different positions viz., $z = 2$, 3 and 4. In Fig. 2 and Fig. 4, the position $z = 2$ has been chosen and the values of R and S have been calculated respectively for different values of n, namely $n = 0$, 0.5 and 1. The variations of stress components with position or with time is quite clear from the graphs.

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